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A UNIFIED THEORY OF ELASTIC DEGRADATION AND DAMAGE BASED ON A LOADING SURFACE

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Abstract—A number of new models with stiffness degradation have been proposd in recent literature in the small strain regime. However, most of these works represent specific formulations, each using its own terminology, notation and assumptions, and relatively little effort has been spent so far towards achieving a common theoretical framework similar for instance to the theory of elastoplasticity. Moreover, most of the existing damage models are presented with intensive recourse to abstract thermodynamics concepts, and they combine stiffness degradation with plasticity, which (though being ultimately necessary to represent the actual material behavior) makes it much more difficult to isolate, analyse and understand the properties of the formulation for elastic stiffness degradation. As a contribution in this field, this paper presents a unifying theoretical framework to describe a class of models for elastic stiffness degradation based on the concept of loading surface. The derivation includes two consecutive steps: first, the constitutive framework for elastic-degrading models with evolution laws which are expressed directly in terms of the secant stiffness (or compliance) tensor, and second the elastic-damage models, in which the scant stiffness (or compliance) is assumed to depend on a reduced set of damage variables with clearer physical meaning and simpler evolution laws. Whenever possible, terminology is borrowed from the classical formulation of elastoplasticity, and thermodynamic concepts are introduced only as needed. Both stress-based and strain-based developments are compared throughout the paper, and the concept of associativity is reanalysed and generalized within the new unified framework of elastic degradation. The most significant scalar damage models found in the literature are reinterpreted in the context of this unified theory. Finally, a general expression is obtained for the tangential stiffness operator of associated scalar models (stress- and strain-based) of the $(1-D)$ type, that includes all the models considered as particular cases. More general damage formulations [scalar non-$(1-D)$, vectorial, tensorial] are reviewed and discussed systematically in a sequel paper.

1. INTRODUCTION

Recent years have seen an increase in the number of papers devoted to material models representing the degradation of the elastic properties or initial material stiffness in the small strain regime. It is well-known how the modulus of elasticity, manifested by loading–unloading–reloading experiments on engineering materials such as concrete, rock, metals, ceramics or composites, degrades progressively when the material is subjected to stresses and strains exceeding some threshold values. Restricting the attention to situations prior to the onset of localization, the behavior of these materials has often been modeled ignoring the fact of stiffness degradation by using for instance *elastoplasticity* (EP) (Chen, 1982; Pramono and Willam, 1989). In the spirit of some previous models developed for soils, some authors also used linear elastic relations with variable moduli for concrete (Kupfer and Grestle, 1973). However, in those works the variable moduli were not a subject of study *per se* but only some values that once introduced in Hooke's law would reproduce the experimental response. Consequently, crucial points for stiffness degradation such as irreversibility, energy dissipation, anisotropy, damage, etc. were not addressed.

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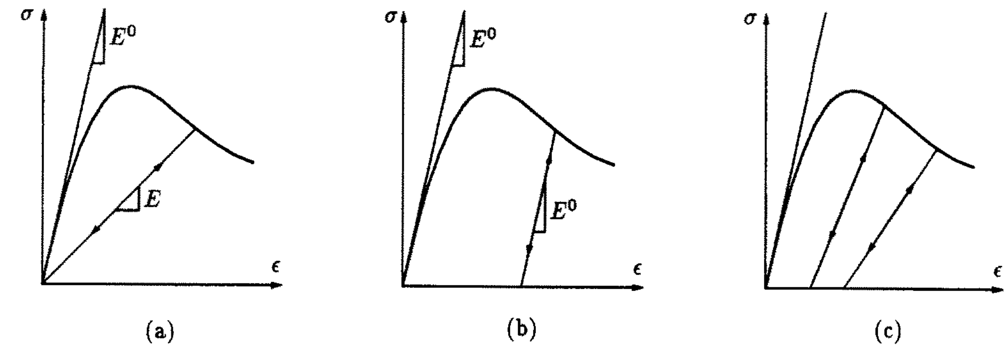

Fig. 1. Typical stress–strain diagrams for: (a) elastic-degrading material; (b) elastic–plastic material; (c) realistic behavior from observation.

In early works on tertiary creep of metals, Kachanov (1958) introduced the concept of uniaxial damage as the reduction of stress-carrying area of the specimen, and its associated reduction of stiffness and concomitant increase of the stresses on the remaining part of the cross-section (effective stresses). In 1976, Dougill formulated a three-dimensional constitutive description for *elastic-fracturing* materials, and proposed some evolution laws for the degradation of stiffness of such material models. However, Dougill's terminology may be misleading since it might indicate a relation to the field of Fracture Mechanics that, although existent, is neither direct nor simple. For this reason, and to avoid unnecessary confusion, Dougill's terminology is not being used in this paper in spite of its historical interest. Instead, the terms *elastic degradation* (ED) and *elastic-degrading models* have been adopted for those non-linear material formulations of the continuum type in which the degradation of stiffness is such that full unloading always leads to the origin [zero stresses, zero permanent strains, see Fig. 1(a) for a simplified uniaxial ED diagram, in contrast with Fig. 1(b) for classical EP with constant initial elastic unloading modulus and non-zero permanent or plastic strain].

It is clear that both EP and ED models are simplifications of the actual behavior observed in experiments, as shown in Fig. 1(c), where both stiffness degradation and irreversible plastic strain take place. This type of material behavior can be obtained by appropriate combination of both types of models, as already proposed by a number of authors (Hueckel and Maier, 1977; Dragon and Mróz, 1979; Bažant and Kim, 1979; Cordebois and Sidoroff, 1982; Ortiz, 1985; Han and Chen, 1986; Simo and Ju, 1987; Chow and Wang, 1987). Rationality and common sense, however, suggest that before proceeding to such combinations, both EP and ED models should be fully understood individually, their main properties and characteristics determined, their general expressions established and the main alternative options for their formulation investigated. In classic elastoplasticity this has been done already years ago, and a well-established and more or less unified theoretical description exists which seems to be accepted by the majority of the scientific and engineering community. But this seems not to be the case of elastic-degrading models. A review of the existing literature in this field clearly indicates a large scatter of terminology and basic assumptions, and the non-existence of a unified theoretical description. On the contrary, in recent years, the proliferation of models based on Continuum Damage Mechanics (CDM) and intensive recourse to abstract thermodynamic concepts has complicated the whole picture even further.

As the consequence of the foregoing arguments, it seems appropriate to devote new efforts to establish a unified theory of elastic degradation that brings ED models to a similar degree of development as their EP counterparts. Also, this will make it possible to consider combinations of plasticity and stiffness degradation based on more solid grounds. The present paper is intended to contribute to this general objective by proposing a general description of ED models which borrows concepts and terminology from the well-known flow theory of plasticity. Some of these concepts have already been used in the existing literature, although not in the systematic manner proposed here. It will be apparent that thermodynamic arguments cannot be avoided completely, especially when concepts such

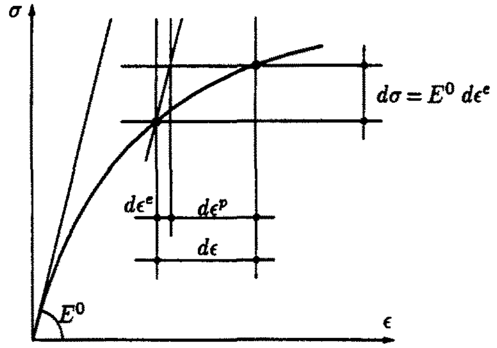

Fig. 2. Definition of the plastic strain rate.

as normality and associativity are generalized to new environments. However, in order to achieve a more intuitive understanding of the subject, their introduction and use has been restricted to the very essentials.

After the Introduction, the paper includes a short review of the stress- and strain-based formulations of classical plasticity, with the objective of establishing a reference framework of concepts, notation and terminology, for better understanding the similar developments of elastic degradation in the subsequent sections. In Section 3, the general formulation of a stress-based elastic-degrading material is presented. It includes expressions for the rate of secant stiffness and the tangent stiffness operator. In Section 4, basic thermodynamic arguments are introduced as well as the concept of associativity in the compliance space. The differences between thermodynamic associativity and the more classical concept of normality in the strain space are presented and discussed in the context of the damage model by Ortiz (1985). The same developments of Sections 3 and 4 are summarized in Section 5 delineating the dual strain-based formulation. The concept of damage variables is introduced in Section 6, as a (usually) reduced set of variables which fully describe the current state of damage and the current value of scant stiffness or compliance. This makes it possible to consider the damage space, to define a damage rule, the corresponding thermodynamic forces and the concept of associativity in this space, and discuss its relation to associativity in the compliance and strain spaces. Section 7 includes some examples of the most significant scalar damage models found in the literature. Disregarding those aspects that are not relevant here (plasticity, positive–negative projection operators, etc.), it is shown that the main formulations currently in the literature can be reinterpreted in the context of this unified theory. In Section 8, a general formulation including a single expression of the tangential stiffness operator is obtained for associated scalar damage models (both stress- and strain-based) of the $(1-D)$ type. A sequel to the current paper contains a similar exercise with other more complex types of damage formulations [scalar non-$(1-D)$, vectorial, tensorial]. Finally, Section 9 summarizes the main conclusions that can be extracted from the work presented in this paper.

2. STRESS- AND STRAIN-BASED FORMULATIONS OF CLASSICAL PLASTICITY

2.1. *Stress-based formulation*

The formulation is built around the assumption of a loading function that can be expressed as $F[\boldsymbol{\sigma}, \mathbf{p}]$, where the brackets "[ ]" enclose the arguments of the function (this convention is used throughout the paper), and $\mathbf{p}$ is a vector of variables determining the current configuration of the loading surface. The rate equations for the stress-based elastoplastic formulation can be expressed as follows (see Fig. 2):

$$\dot{\sigma}_{ij} = E^0_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) \tag{1}$$

$$\dot{\varepsilon}^p_{kl} = \dot{\lambda} m_{kl} \tag{2}$$

$$\dot{F} = \left.\frac{\partial F}{\partial \sigma_{ij}}\right|_{\mathbf{p}} \dot{\sigma}_{ij} + \left.\frac{\partial F}{\partial p_k}\right|_{\boldsymbol{\sigma}} \dot{p}_k = 0 \tag{3}$$

where E^0_{ijkl} are the components of the elastic stiffness tensor $\mathbf{E}^0$, the plastic multipler $\dot{\lambda}$ defines the magnitude of the plastic strain rate, m_{kl} specifies its direction (flow rule, which is normally expressed in terms of a plastic potential Q as $m_{kl} = \partial Q/\partial\sigma_{kl}$) and subscripts $\mathbf{p}$ and $\boldsymbol{\sigma}$ in (3) indicate that the partial derivatives are calculated for constant $\mathbf{p}$ and constant $\boldsymbol{\sigma}$ respectively (this convention is used throughout the paper). Equation (3) represents the linearized form (truncated Taylor expansion) of the consistency condition, meaning that during plastic loading the current stress state always remains on the current loading surface. Assuming that the parameters p_i are functions of the plastic strain ε^p_{kl}, i.e. $\dot{p}_i = (\partial p_i/\partial \varepsilon^p_{kl})\dot{\varepsilon}^p_{kl}$, and using (2), one can rewrite (3) as

$$n_{ij}\dot{\sigma}_{ij} - H\dot{\lambda} = 0 \quad \text{with} \quad n_{ij} = \left.\frac{\partial F}{\partial \sigma_{ij}}\right|_{\lambda} \quad \text{and} \quad H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\boldsymbol{\sigma}} = -\left.\frac{\partial F}{\partial p_i}\right|_{\boldsymbol{\sigma}} \frac{\partial p_i}{\partial \varepsilon^p_{kl}} m_{kl}. \tag{4a, b, c}$$

In these expressions, n_{ij} involves derivatives of F for constant values of the plastic multiplier λ (i.e. $\dot{\lambda} = \dot{\varepsilon}^p_{kl} = 0$, no change of plastic strain and internal variables p_i), with the geometric meaning of the direction normal to the current loading surface $F = 0$ in stress space. The hardening–softening modulus H involves derivatives for constant values of $\boldsymbol{\sigma}$. H is positive in the hardening regime with an expanding loading surface, zero for perfect plasticity with a fixed loading surface and negative in the softening regime with a contracting surface. The formulation is called associated whenever the loading function F and the flow rule are defined in such a way that n_{ij} and m_{ij} are fully proportional (often stated alternatively as $Q = F$). The loading–unloading criterion can be given as two unilateral restrictions in terms of the loading function (that must be negative or zero, but not positive) and the plastic multiplier (that must be positive or zero, but not negative). This is normally expressed by the following three conditions: (i) $F \leqslant 0$, (ii) $\dot{\lambda} \geqslant 0$, and (iii) $F\dot{\lambda} = 0$.

Once the "directions" $\mathbf{n}$ and $\mathbf{m}$ are established, the combination of eqns (1), (2) and (4a) leads to the classical strain-driven format of the plastic multiplier:

$$\dot{\lambda} = \frac{n_{cd}E^0_{cdkl}\dot{\varepsilon}_{kl}}{H + n_{pq}E^0_{pqrs}m_{rs}}. \tag{5}$$

In the numerator, one can identify the trial stress increment $\dot{\sigma}^t_{cd} = E^0_{cdkl}\dot{\varepsilon}_{kl}$, and its scalar product with the normal n_{cd}, which gives a positive quantity when $\dot{\sigma}^t_{cd}$ points outside the loading surface (plastic loading) and negative when it points inwards (elastic unloading). Consistency with the sign of $\dot{\lambda}$ requires the denominator in eqn (5) to remain always positive. Since one can in general accept the restriction that F and $\mathbf{m}$ are defined in such a way that the bilinear from $n_{pq}E^0_{pqrs}m_{rs}$ remains positive, this means that there is a limiting value for the softening (negative) modulus H, i.e. $H > -n_{pq}E^0_{pqrs}m_{rs}$.

By introducing eqn (5) into (2) and (1), one obtains the elastoplastic tangential stiffness tensor for plastic loading:

$$\dot{\sigma}_{ij} = E^{\mathrm{t}}_{ijkl}\dot{\varepsilon}_{kl}; \quad E^{\mathrm{t}}_{ijkl} = E^0_{ijkl} - \frac{E^0_{ijab}m_{ab}n_{cd}E^0_{cdkl}}{H + n_{pq}E^0_{pqrs}m_{rs}}. \tag{6a, b}$$

Alternatively, one can express the plastic multiplier in terms of the stress rate directly from eqn (4a)

$$\dot{\lambda} = \frac{1}{H} n_{rs} \dot{\sigma}_{rs}. \tag{7}$$

If this stress-driven format of $\dot{\lambda}$ is introduced into eqn (2), the following expression is obtained from (1) for the tangent compliance:

$$\dot{\varepsilon}_{ij} = C^{t}_{ijkl} \dot{\sigma}_{kl}; \quad C^{t}_{ijkl} = C^{0}_{ijkl} + \frac{1}{H} m_{ij} n_{kl}. \tag{8a, b}$$

Instead of starting from eqn (7), eqns (8) can also be obtained by direct inversion of (6), by using the Shermann–Morrison formula (Shermann and Morrison, 1950). Relations (7) and (8) are much simpler than their strain-driven counterparts (5) and (6), but they exhibit the important limitation that they can only be used in the hardening regime, with positive H. Note that for associated plasticity, indices i, j and k, l in expressions (6) and (8) become interchangeable, reflecting major symmetry in the tangential stiffness and compliance tensors.

2.2. *Strain-based formulation*

In this case, the loading function is expressed as a function of strains and the internal variables $F[\boldsymbol{\varepsilon}, \bar{\mathbf{p}}]$. The rate equations are:

$$\dot{\varepsilon}_{ij} = C^{0}_{ijkl}(\dot{\sigma}_{kl} - \dot{\sigma}^{p}_{kl}) \quad \text{with} \quad \dot{\sigma}^{p}_{kl} = -E^{0}_{klpq} \dot{\varepsilon}^{p}_{pq} \tag{9a, b}$$

$$\dot{\sigma}^{p}_{kl} = \dot{\lambda} \bar{m}_{kl} \tag{10}$$

$$\dot{F} = \bar{n}_{ij} \dot{\varepsilon}_{ij} - \bar{H} \dot{\lambda} = 0 \quad \text{with} \quad \bar{n}_{ij} = \left.\frac{\partial F}{\partial \varepsilon_{ij}}\right|_{\lambda} \quad \text{and} \quad \bar{H} = -\left.\frac{\partial F}{\partial \lambda}\right|_{\varepsilon} = -\left.\frac{\partial F}{\partial \bar{p}_i}\right|_{\varepsilon} \frac{\partial \bar{p}_i}{\partial \sigma^{p}_{kl}} m_{kl}. \tag{11a, b, c}$$

Note that eqns (9a), (10) and (11a) are analogous to eqns (1), (2) and (4a) by interchanging $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, $\mathbf{E}^0$ and $\mathbf{C}^0$, and replacing n_{ij} by $\bar{n}_{ij}$, m_{ij} by $\bar{m}_{ij}$ and H by $\bar{H}$. The subsequent derivation is completely analogous to the previous case, leading to the following stress-driven expressions for the plastic multiplier and the tangential compliance

$$\dot{\lambda} = \frac{\bar{n}_{cd} C^{0}_{cdkl} \dot{\sigma}_{kl}}{\bar{H} + \bar{n}_{pq} C^{0}_{pqrs} \bar{m}_{rs}}; \quad C^{t}_{ijkl} = C^{0}_{ijkl} - \frac{C^{0}_{ijab} \bar{m}_{ab} \bar{n}_{cd} C^{0}_{cdkl}}{\bar{H} + \bar{n}_{pq} C^{0}_{pqrs} \bar{m}_{rs}}. \tag{12a, b}$$

Similarly as before, one may alternatively evaluate the plastic multiplier directly from eqn (11a) in terms of the strain rate, and using (10) and (1), obtain the expression of the tangential stiffness

$$\dot{\lambda} = \frac{1}{\bar{H}} \bar{n}_{rs} \dot{\varepsilon}_{rs}; \quad E^{t}_{ijkl} = E^{0}_{ijkl} + \frac{1}{\bar{H}} \bar{m}_{ij} \bar{n}_{kl}. \tag{13a, b}$$

In this case, the equation for the compliance is more complicated and it can only be used in the hardening range. Equations (13) have both advantages of simplicity and softening capability. In strain-based formulations, $\bar{H}$ is always positive, the strain space formulation of $F = 0$ must always expand in both hardening and softening ranges.

According to the previous equations, for each strain-based formulation one can consider the corresponding dual stress-based formulation which is equivalent. The loading functions of equivalent dual formulations have in general different expressions in terms of stress or strain, but both give the same value when evaluated for a stress state (and its internal variables) or for its corresponding strain state (and its internal variables). Under these conditions, and assuming that F represents the numerical value of the loading function (and not the particular mathematical expression), the same symbol F can be used in both dual formulations. The flow rules of equivalent dual formulations $\bar{m}_{ij}$ and m_{ij} can be related to each other. This relation is easily obtained by introducing eqn (2) and its counterpart (10) in (9b), and eliminating the multiplier $\dot{\lambda}$ from both sides. Realizing that their definitions (4b) and (11b) involve derivatives for constant λ, then $\bar{n}_{ij}$ and n_{ij} can also be related through the initial elastic stiffness E^0_{ijkl}. Finally, $\bar{H}$ can be related to H by enforcing the same $\dot{\lambda}$ for the dual formulations. The resulting relations are:

$$\bar{m}_{ij} = -E^0_{ijkl} m_{kl} \quad \text{or} \quad m_{ij} = -C^0_{ijkl} \bar{m}_{kl} \tag{14a, b}$$

$$\bar{n}_{ij} = E^0_{ijkl} n_{kl} \quad \text{or} \quad n_{ij} = C^0_{ijkl} \bar{n}_{kl} \tag{14c, d}$$

$$\bar{H} = H + n_{ij} E^0_{ijkl} m_{kl} \quad \text{or} \quad H = \bar{H} + \bar{n}_{ij} C^0_{ijkl} \bar{m}_{kl}. \tag{14e, f}$$

With these expressions, (13a, b) can be easily converted into eqns (5) and (6b), and (12a, b) into (7) and (8b). Also, (14e) shows that the condition that $\bar{H}$ be always positive is fully consistent with the analogous condition that the denominator of (5) for stress-based models remains also positive. In other terms, the strain-driven format of stress-based elastoplasticity coincides one-to-one with the strain-driven format of strain-based elastoplasticity.

3. STRESS-BASED FORMULATION OF ELASTIC-DEGRADING MATERIAL. ASSOCIATIVITY IN THE STRAIN SPACE

3.1. *Basic equations of elastic-degrading materials*

The characteristic feature of an elastic-degrading material is the existence of a total stress–strain relationship

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = C_{ijkl} \sigma_{kl} \tag{15a, b}$$

where E_{ijkl} and C_{ijkl} are the components of the secant stiffness and compliance tensors **E** and **C**, which vary during the loading process. During unloading–reloading, the material stiffness is assumed to remain constant and equal to its current secant value. Under these conditions, the elastic-degrading material can be considered equivalent to an anisotropic elastic material. The usual requirements of major symmetry for the anisotropic elasticity tensor (based on the requirement of the existence of an energy potential, or alternatively, on the requirement of zero energy dissipation upon a closed excursion in the strain or stress space) also apply in this case, and therefore **E** and **C** must remain symmetric during their entire evolution.

The fact that unloading–reloading (secant) stiffness is assumed constant regardless of the values of strains and stresses implies that no microcrack closure–reopening effects are considered. It is well-known that for many materials the reduction of stiffness is caused by the formation and propagation of tensile microcracks. Upon load reversal, these microcracks tend to close and the overall stiffness increases again approaching its initial value. However, taking into account this effect under general multiaxial conditions is not trivial. Microcracks do not heal during closure (i.e. new tensile stresses would reopen the same microcracks) and therefore the closure effect should not reverse the energy dissipated during the cracking process. This means that a formulation for closure–reopening within the elastic region inside the loading surface should be of a non-dissipative type, with a well-defined energy potential. A careful examination of the existing literature, mostly based on the so-called P^+ and P^- projection operators and the corresponding decomposition of the stress (or strain) tensor into a positive and a negative part (Mazars and Lemaitre, 1984; Ortiz, 1985; Simo and Ju, 1987; Yazdani and Schreyer, 1988; Ju, 1989; Mazars and Pijaudier-Cabot, 1989; Chaboche, 1990), shows that under certain closed-cycle stress (or strain) histories involving rotation of principal directions these procedures may not satisfy the previous requirements, leading to net energy dissipation (or, reversing the path, net energy generation!), that seems not acceptable for a general formulation (Carol and Willam, 1994)

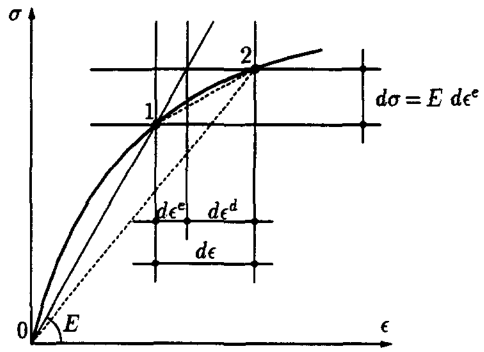

Fig. 3. Definition of the degrading strain rate.

the formulation of microcrack closure–reopening effects has been considered a matter of on-going research and left out of the scope of the present paper.

Differentiation of (15) leads to

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{\varepsilon}_{kl} + \dot{E}_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \dot{\varepsilon}_{ij} = C_{ijkl}\dot{\sigma}_{kl} + \dot{C}_{ijkl}\sigma_{kl}. \tag{16a, b}$$

The rates of change of both stiffness and compliance can be related to each other. By definition, they are inverse to each other, i.e. $\mathbf{C}:\mathbf{E} = \mathbf{I}_4$ must yield the fourth-order identity tensor. Assuming sufficient continuity, this equation may be differentiated as $\dot{\mathbf{C}}:\mathbf{E}+\mathbf{C}:\dot{\mathbf{E}} = \mathbf{0}$, and multiplied by $\mathbf{E}$ on the left or $\mathbf{C}$ on the right, yielding the relations between the stiffness and compliance changes

$$\dot{E}_{ijkl} = -E_{ijpq}\dot{C}_{pqrs}E_{rskl} \quad \text{or} \quad \dot{C}_{ijkl} = -C_{ijpq}\dot{E}_{pqrs}C_{rskl}. \tag{17a, b}$$

3.2. *Rate equations based on a flow rule for degrading strain. Associativity in strain space*

As shown by several authors (Hueckel and Maier, 1977; Ortiz, 1985; Yazdani and Schreyer, 1988), in a first stage of development, the elastic-degrading material based on a loading surface can be formulated in a very similar way to classical plasticity. The loading function $F[\boldsymbol{\sigma}, \mathbf{p}]$ is defined in this case as the criterion for distinguishing the cases of unloading, where no changes in stiffness properties occur (postulating elastic unloading–reloading with the current secant stiffness), and new virgin loading, where stiffness degradation progresses accompanied by *degrading strain*. In Fig. 3, the main variables involved in the formulation of a differential increment of virgin loading are represented on a one-dimensional stress–strain diagram. In the figure, the total strain and stress increments are $\dot{\varepsilon}$ and $\dot{\sigma}$. The total strain increment $\dot{\varepsilon}$ can be decomposed into elastic and degrading parts. The elastic part $\dot{\varepsilon}^e$ is defined as the strain that, with the current secant stiffness E, would produce the same stress increment $\dot{\sigma}$, and the degrading strain increment is defined as $\dot{\varepsilon}^d = \dot{\varepsilon} - \dot{\varepsilon}^e$.

In this context, and postponing for a few paragraphs the use of the equations given in the previous section, the following rate equations can be considered

$$\dot{\sigma}_{ij} = E_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^d_{kl}) \tag{18}$$

$$\dot{\varepsilon}^d_{kl} = \dot{\lambda} m_{kl} \tag{19}$$

$$\dot{F} = n_{ij}\dot{\sigma}_{ij} - H\dot{\lambda} = 0 \quad \text{with} \quad n_{ij} = \left.\frac{\partial F}{\partial \sigma_{ij}}\right|_{\lambda} \quad \text{and} \quad H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\sigma} = -\left.\frac{\partial F}{\partial p_i}\right|_{\sigma}\frac{\partial p_i}{\partial \varepsilon^d_{kl}} m_{kl}. \tag{20a, b, c}$$

This set of equations looks practically identical to its counterpart in classical plasticity [eqns (1), (2) and (4)], except for the secant instead of the initial stiffness, and a different physical meaning of the degrading strain if compared with the plastic strain and of its corresponding flow rule (here the excess over the "elastic" strain is associated with the current secant modulus instead of the initial modulus). Nevertheless, independently of the new physical meaning, these equations can be combined in the same way as the ones in plasticity, and the following expressions can be obtained for the degrading multiplier $\dot{\lambda}$ and the tangential stiffness

$$\dot{\lambda} = \frac{n_{cd}E_{cdkl}\dot{\varepsilon}_{kl}}{H+n_{pq}E_{pqrs}m_{rs}}; \quad E^{\mathrm{t}}_{ijkl} = E_{ijkl} - \frac{E_{ijab}m_{ab}n_{cd}E_{cdkl}}{H+n_{pq}E_{pqrs}m_{rs}}. \tag{21a, b}$$

Similarly to stress-based plasticity, alternative expressions based on compliance, analogous to (7) and (8) can be derived, although with the important restriction that they are only valid in the hardening regime, with positive H. The concept of associativity *in the strain space* (when **m** is defined parallel to **n**) can also be established for elastic-degrading materials. Also in this case (and provided the secant stiffness and compliance tensors satisfy major symmetry during their evolution, as stated in the previous section), associativity means that pairs of indices i,j and k,j in (21b) become interchangeable and therefore the tangential stiffness tensor exhibits major symmetry.

3.3. *Degradation rule and its relation to the flow rule*

In spite of their formal similarity, however, eqns (21) show an important qualitative difference with their plastic counterparts (5) and (6). The fact that the variable secant stiffness is involved in the expressions means that the functions and parameters usually defined in plasticity (expressions for F, the hardening–softening laws and the flow rule), are not sufficient to define the evolution of the degradation model and to integrate eqns (21), because we need an additional law for the secant stiffness tensor **E** itself. In Fig. 3 one can see that degrading strain of an elastic-degrading model is intrinsically associated with the degradation of the secant stiffness. In fact, it looks as if once the amount of degrading strain has been determined, the degradation of stiffness could follow automatically. However, the representation in the figure can be misleading since each axis is a one-dimensional representation of a tensorial quantity. A more realistic picture of the problem can be achieved by taking advantage of minor symmetries and considering six-component vectors for stress and strain, and a symmetric 6×6 matrix for the stiffness tensor. If the current stress–strain state is known and one tries to calculate the secant stiffness matrix, one has only six equations for a total of 21 unknowns. Therefore, there are many possible secant stiffness matrices which would lead to the same values of total stress and strain.

In the same way, the flow rule defined by (19) in terms of the degrading strains (a second-order tensor), does not contain enough information so as to define uniquely the variation of the secant stiffness (a fourth-order tensor). In fact, one could consider an infinite number of different secant stiffnesses which could result in the same flow rule in the strain space. To address this aspect, one should compare the two rate equations (16a) and (18), and obtain

$$E_{ijkl}\dot{\varepsilon}^{\mathrm{d}}_{kl} = -\dot{E}_{ijkl}\varepsilon_{kl}. \tag{22}$$

Equation (17a) can now be introduced in the right-hand side of (22), both sides can be multiplied by C_{pqij}, and using (15a) one obtains (Ortiz, 1985; Yazdani and Schreyer, 1988)

$$\dot{\varepsilon}^{\mathrm{d}}_{pq} = \dot{C}_{pqrs}\sigma_{rs}. \tag{23}$$

This equation, that can be alternatively obtained by inversion of (18) and comparison to (16b), indicates a relationship between the increment of secant compliance and the increment of degrading strain. When one is known, the other follows (but not the opposite, in accordance to the previous discussion). Now, it is convenient to define a "generalized flow rule" or *degradation rule* for the secant compliance [as suggested previously by Ortiz (1985), although he used different terminology and notation]

$$\dot{C}_{ijkl} = \dot{\lambda} M_{ijkl} \tag{24}$$

where $\dot{\lambda}$ defines the magnitude and **M** the direction of the rate of change of **C**. By replacing (24) and (19) in (23), and eliminating the multipliers from both sides (this can be done since by definition the flow rule is only a direction, and both **m** and **M** can be re-scaled conveniently), one finally obtains

$$m_{ij} = M_{ijkl}\sigma_{kl}. \tag{25}$$

This equation is in essence equivalent to eqn (2.7) in Hueckel and Maier (1977), and to eqn (3.36) in Ortiz (1985). It indicates that once the degradation rule **M** has been established, the corresponding flow rule for degrading strains, **m**, follows automatically. The symmetry condition in Section 3.1 of **C** requires that the degradation rule **M** also be symmetric. Associativity *in the strain space* can still be considered when the degradation rule is defined in such a way that the resulting **m** given by (25) is parallel to $\mathbf{n} = \partial F/\partial \boldsymbol{\sigma}$.

3.4. *Final equations for the elastic-degrading material*

Summarizing, the constitutive description of elastic degradation becomes complete after definition of F, the hardening–softening law and the fourth-order degradation rule **M** (instead of the second-order flow rule **m** in plasticity). The final expressions for the tangent stiffness tensor and for the evolution of the secant stiffness, valid in both hardening and softening regimes, are obtained by direct substitution of (25) into (21b), and into (21a), (24) and (17a):

$$E^{t}_{ijkl} = E_{ijkl} - \frac{E_{ijab}M_{abxy}\sigma_{xy}n_{cd}E_{cdkl}}{H + n_{pq}E_{pqrs}M_{rsuv}\sigma_{uv}} \tag{26}$$

$$\dot{E}_{ijkl} = -E_{ijpq}\dot{C}_{pqrs}E_{rskl}; \quad \dot{C}_{ijkl} = M_{ijkl}\frac{n_{ab}E_{abcd}\dot{\varepsilon}_{cd}}{H + n_{pq}E_{pqrs}M_{rsuv}\sigma_{uv}}. \tag{27a, b}$$

Note that the equations strictly necessary for integrating the model reduce in fact to (27a, b), since once the secant compliance or stiffness are updated, the total stress can be directly obtained from total strains via (15a, b). The tangential expression (26) is, however, convenient, and strictly necessary for the study of incremental solutions and the analysis of failure diagnostics based on the tangential stiffness and the corresponding acoustic tensor (Rizzi *et al.*, 1993).

4. THERMODYNAMIC CONSIDERATIONS AND ASSOCIATIVITY IN THE COMPLIANCE SPACE

4.1. *Free energy, degrading dissipation and thermodynamic forces in the compliance space*

Further insight into the formulation of elastic-degrading material requires the introduction of some thermodynamic concepts. The basic expression is that of the mechanical free energy of the system, i.e. the energy that is "stored" in the material and can be recovered upon unloading. For the elastic-degrading material, this energy is given by the elastic energy corresponding to the current secant stiffness (or compliance) and can be expressed (per unit volume) as:

$$w = \tfrac{1}{2}\varepsilon_{ij}E_{ijkl}\varepsilon_{kl} \quad \text{or} \quad w = \tfrac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl}. \tag{28a, b}$$

Focusing on the first of these equations, the rate of variation of the free energy due to variations of strains and secant stiffness is:

$$\dot{w} = \left.\frac{\partial w}{\partial \varepsilon_{kl}}\right|_{\mathbf{E}} \dot{\varepsilon}_{kl} + \left.\frac{\partial w}{\partial E_{ijkl}}\right|_{\varepsilon} \dot{E}_{ijkl} \quad \text{or} \quad \dot{w} = \sigma_{kl}\dot{\varepsilon}_{kl} + \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{kl}\dot{E}_{ijkl}. \tag{29a, b}$$

The first term on the right-hand side of (29b) has the meaning of the external work supplied to the system by the applied stresses (with constant stiffness), while the second term corresponds to the dissipation of energy due to the stiffness degradation for constant strain (i.e. with no external work supply). This second term, taken with negative sign (the variation of free energy due to the degradation of stiffness is intrinsically negative, but it is convenient to define dissipation as a positive quantity) is called *rate of degrading dissipation*, $\dot{d}$. In the absence of heat transfer, eqn (29b) states the balance of energy and work in the material: the elastic energy accumulated is equal to the external supply of work less than the dissipation due to the elastic degradation process. The rate of degrading dissipation can be expressed as in (29b), or by using (17) and (15), in terms of stresses and compliance:

$$\dot{d} = -\tfrac{1}{2}\varepsilon_{ij}\dot{E}_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \dot{d} = \tfrac{1}{2}\sigma_{ij}\dot{C}_{ijkl}\sigma_{kl}. \tag{30a, b}$$

It is natural now to introduce the concept of a *thermodynamic or generalized force* $-\bar{Y}_{ijkl}$ which is conjugate to the secant stiffness E_{ijkl} and, similarly, $-Y_{ijkl}$ conjugate to the secant compliance C_{ijkl}, as the quantities that yield the rate of degrading dissipation when multiplied by the rates $\dot{E}_{ijkl}$ and $\dot{C}_{ijkl}$ respectively

$$(-\bar{Y}_{ijkl})\dot{E}_{ijkl} = \dot{d} \quad \text{and} \quad (-Y_{ijkl})\dot{C}_{ijkl} = \dot{d}. \tag{31a, b}$$

The negative signs are included to follow the usual notation in the literature (in accordance to the change of sign assumed earlier for $\dot{d}$). From (30) and (31) it follows to identify

$$-\bar{Y}_{ijkl} = -\tfrac{1}{2}\varepsilon_{ij}\varepsilon_{kl} \quad \text{and} \quad -Y_{ijkl} = \tfrac{1}{2}\sigma_{ij}\sigma_{kl}. \tag{32a, b}$$

Note that (28a, b) can also be considered as specific expressions of a general free energy potential $w[\boldsymbol{\varepsilon}, \mathbf{E}]$ (or $w[\boldsymbol{\sigma}, \mathbf{C}]$) with internal variables E_{ijkl} (or C_{ijkl}), and then the former definitions of $-\bar{Y}_{ijkl}$ and $-Y_{ijkl}$ are equivalent to

$$\bar{Y}_{ijkl} = \left.\frac{\partial w}{\partial E_{ijkl}}\right|_{\varepsilon} \quad \text{and} \quad Y_{ijkl} = \left.\frac{\partial w}{\partial C_{ijkl}}\right|_{\varepsilon} = -\left.\frac{\partial w}{\partial C_{ijkl}}\right|_{\sigma}. \tag{33a, b}$$

Note also that the same potential w is used for the definitions of both stress- and strain-based thermodynamic forces. This is possible in this case because of assumption (28a, b), that implies linear behavior with a secant stiffness which remains constant upon unloading. In a more general context with non-linear unloading–reloading, the use of two different potentials, the elastic strain energy and its complementary would be required (in a more general thermomechanical environment these potentials would correspond to the Helmholtz and Gibbs potentials).

4.2. *Irreversibility of degrading dissipation*

The fact that degradation of stiffness is an irreversible process implies that the degrading dissipation $\dot{d}$ must be a non-negative quantity according to the second principle of thermodynamics (Malvern, 1969). Using (30b) and replacing $\dot{C}_{ijkl}$ according to the degradation rule (24), this condition can be expressed as

$$\dot{d} = \tfrac{1}{2}\sigma_{ij}\sigma_{kl}M_{ijkl}\dot{\lambda} \geqslant 0. \tag{34}$$

Since $\dot{\lambda} \geqslant 0$ (otherwise we have unloading with $\dot{\lambda} = \dot{d} = 0$), a sufficient condition for (34) to be satisfied is that M_{ijkl} be a positive definite tensor. This is a second requirement for the degradation rule in addition to being symmetric as stated in Section 3.3. Note also that, using

(23), the expression for degrading dissipation can be rewritten in terms of the degrading strain as

$$\dot{d} = \tfrac{1}{2}\sigma_{ij}\dot{\varepsilon}^{\mathrm{d}}_{ij}; \quad \dot{d} = (-y_{ij})\dot{\varepsilon}^{\mathrm{d}}_{ij} \quad \text{with} \quad -y_{ij} = \tfrac{1}{2}\sigma_{ij}. \qquad (35\mathrm{a,b,c})$$

In Fig. 3, $\dot{d}$ corresponds to the area enclosed in the triangle 0, 1, 2, which is the difference between the initial w plus the external work, minus the new w, as one should have expected. Equations (35) also indicate that the thermodynamic force conjugate to the degrading strain, $-y_{ij}$, is equal to the stress tensor divided by two. By replacing $\dot{\varepsilon}^{\mathrm{d}}_{ij} = \dot{\lambda} m_{ij}$ from (19), the restriction $\dot{d} \geqslant 0$ can be alternatively expressed in terms of the flow rule in strain space as $\sigma_{ij} m_{ij} \geqslant 0$, which is analogous to the plastic dissipation inequality in classical plasticity.

4.3. $\partial F/\partial(-\mathbf{Y})$ *and associativity in the compliance space*

Having established a generalized flow rule for the increments of the compliance tensor C_{ijkl} and its corresponding generalized forces $-Y_{ijkl}$, it is natural to cast normality and associativity in the (fourth-order) compliance space. In fact, these concepts follow naturally from the previous definitions. The generalization of the normal to the loading surface $n_{ij} = \partial F/\partial \sigma_{ij}$ can be established as:

$$N_{ijkl} = \left.\frac{\partial F}{\partial(-Y_{ijkl})}\right|_{\lambda}. \qquad (36)$$

This derivative and N_{ijkl} itself are only defined when F is expressed as a function of the thermodynamic force $-Y_{ijkl} = \sigma_{ij}\sigma_{kl}/2$, which is the case for most of the stress-based loading functions used in the existing literature. With $\mathbf{N}$ and $\mathbf{M}$, the concept of *associativity in the compliance space* can be defined as a condition of proportionality between them (i.e. when $\mathbf{N}$ exists and $N_{ijkl} = kM_{ijkl}$).

If the loading function F is such that (36) exists, then the relation between n_{ij} and N_{ijkl} can be derived easily from the definition of n_{ij}. By using the chain differentiation:

$$n_{ij} = \frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial(-Y_{pqrs})}\frac{\partial(-Y_{pqrs})}{\partial \sigma_{ij}} \qquad (37)$$

where all partial derivatives are with λ = constant, and $\partial(-Y_{pqrs})/\partial\sigma_{ij}$ can be obtained from (32b), yielding

$$n_{ij} = N_{ijkl}\sigma_{kl}. \qquad (38)$$

4.4. *Relation between associativity in the compliance and strain spaces*

Equation (38), together with the analogous relation derived for the flow rule m_{ij} (25), allows us to establish that *associativity in the compliance space implies associativity in the strain space*. The proof is straight forward: if the formulation is associated in the compliance space, N_{ijkl} will be parallel to M_{ijkl}; according to (38) and (25), this implies also that n_{ij} will be parallel to m_{ij}.

The previous relation, however, does not hold in the opposite sense, i.e. an ED model can be associated in the strain space but at the same time be non-associated in the compliance space. This is always the case when $n_{ij} = m_{ij}$ but at the same time *F is not a function of* $-\mathbf{Y}$ and therefore $\mathbf{N}$ is not defined. When *F can be expressed* as a function of $-\mathbf{Y}$ (and therefore $\mathbf{N}$ exists), the same situation is also possible. To illustrate this, consider some specific loading function F^* and the corresponding n^*_{ij} and N^*_{ijkl} (which are also fixed uniquely since they are derivatives of F^*). Assume that a flow rule for strains $m_{ij} = m^*_{ij} = n^*_{ij}$ is chosen. With this choice, the model will be associated in the strain space. Now, according to the discussion in the previous section, there are many different flow rules for the compliance M_{ijkl} that can lead through (25) to the particular m^*_{ij} chosen. However, if all of them are different, only one of them will satisfy $M_{ijkl} = N^*_{ijkl}$. That specific formulation will be associated in both compliance and strain spaces, and all other descriptions with $M_{ijlk} \neq N^*_{ijkl}$ which satisfy $m_{ij} = M_{ijkl}\sigma_{kl} = n^*_{ij}$, will be associated in strain space but not in the compliance space.

A clear example of this case can be extracted from the damage model by Ortiz (1985). Simplifying the constitutive formulation presented in that paper in order to keep only the essential ingredients related to the present discussion (considering only the mortar model, no plastic strains and no positive or negative projection operators, which corresponds to the case with all stress eigenvalues of the same sign), the model implies the following degradation rule for compliance and the following loading function [eqns (3.40)–(3.43) in the paper]

$$M_{ijkl} = \frac{\sigma_{ij}\sigma_{kl}}{\sigma_{pq}\sigma_{pq}}; \quad F = \tfrac{1}{2}\sigma_{pq}\sigma_{pq} - A. \tag{39a, b}$$

By taking derivatives of this expression of F with respect to $(-Y_{ijkl}) = \sigma_{ij}\sigma_{kl}/2$, one obtains [eqn (36)] that in this case $N_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$. This is not the same as the expression for M_{ijkl} above, and therefore the model *is not* associated in the compliance space. However, by using (38) and (25) one can calculate $n_{ij} = N_{ijkl}\sigma_{kl} = \sigma_{ij}$ and $m_{ij} = M_{ijkl}\sigma_{kl} = \sigma_{ij}$ which indicates that the model *is* associated in the strain space, as stated by the author. Note that a model similar to Ortiz's, which preserves associativity at both levels can be obtained by redefining F such that its derivatives would give $N_{ijkl} = M_{ijkl}$, or alternatively (and in a simpler way) by redefining $M_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$.

The facts that associativity appears at two different levels, and that associativity at the compliance level implies associativity at the strain level but not vice versa, raise new intriguing questions regarding properties normally accepted for associated elastoplasticity, and their extension to elastic degradation.

The first property normally linked to associativity is the symmetry of the tangential stiffness tensor. Accepting the restriction that M_{ijkl} is symmetric (and therefore the secant E_{ijkl} also remains symmetric), eqn (21b) indicates that symmetry of the tangent operator is linked to the condition $m_{ij} = n_{ij}$, and therefore it only requires associativity at strain level (i.e. even if the model is not associated at compliance level, the tangent operator is still symmetric). In fact, this is not too surprising because the tangent operator relates increments of strain and increments of stress (its corresponding thermodynamic force) and therefore only the associativity at this level could be expected to intervene. In order to observe some type of symmetry related to the associativity at the compliance level, one would have to obtain the incremental relation between the compliance tensor C_{ijkl} and its corresponding thermodynamic force, $-Y_{ijkl}$. To this end, one can reformulate the consistency condition (20) in terms of the increments of thermodynamic force $-\dot{Y}_{ijkl}$ instead of stress. This leads to

$$\dot{F} = N_{ijkl}(-\dot{Y}_{ijkl}) - H\dot{\lambda} = 0. \tag{40}$$

Note that since $-\dot{Y}_{ijkl} = \dot{\sigma}_{pq}\partial(-Y_{ijkl})/\partial\sigma_{pq}$ and $N_{ijkl}\partial(-Y_{ijkl})/\partial\sigma_{pq} = n_{pq}$, (40) is in fact equivalent to (20), and therefore H is the same in both equations. From (40), one can isolate $\dot{\lambda}$ and substitute it into (24), obtaining

$$\dot{C}_{ijkl} = A_{ijklpqrs}(-\dot{Y}_{pqrs}) \quad \text{where} \quad A_{ijklpqrs} = \frac{1}{H} M_{ijkl} N_{pqrs}. \tag{41a, b}$$

This equation shows that the eighth-order tensor $\mathbf{A}$ relating the increments of compliance and its thermodynamic force exhibits major symmetry only when $M_{ijkl} = N_{ijkl}$, i.e. when the model is associated at the compliance level.

A second property traditionally related to associativity in elastoplastic literature is that an associated flow rule maximizes the plastic dissipation (Hill, 1950). One of the simplest derivations of this property is based on the notion of the plastic Lagrangian $\mathscr{L}^p = -\sigma_{ij}$

$\dot{\varepsilon}^p_{ij}+\dot{\lambda}F$ which, for stationary conditions with respect to variations of stresses (Steinmann, 1992) yields $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial F/\partial\sigma_{ij}$. For ED formulations, one must consider the Lagrangian of elastic degradation (31b), that is stationary with respect to the variation of the thermodynamic force $-Y_{ijkl}$, yielding

$$\mathscr{L}^d = Y_{ijkl}\dot{C}_{ijkl}+\dot{\lambda}F \quad \text{and} \quad \dot{C}_{ijkl} = \dot{\lambda}\frac{\partial F}{\partial(-Y_{ijkl})}. \tag{42a, b}$$

This means that associativity at compliance level maximizes degrading dissipation.

An apparent contradiction is obtained if the same argument is applied to the dissipation based on the degrading strains (35), instead of compliance (31b). In that case, the corresponding Lagrangian, which reads $\mathscr{L}^d = -\sigma_{ij}\dot{\varepsilon}^d_{kl}/2+\dot{\lambda}F$, yields $\dot{\varepsilon}^d_{ij} = \dot{\lambda}\partial F/\partial\sigma_{ij}$ when made stationary with respect to variations of stress and after incorporating the factor 1/2 into the multiplier λ. This means that degrading dissipation is also maximized when the model is associated at the strain level. This is in fact not consistent with the previous paragraph, since one can consider a model which is associated at the strain level but not at the compliance level. The solution to this apparent contradiction resides on the different character of the derivatives used to impose the stationary condition. In the case of the compliance (42), the derivatives are for constant $\dot{C}_{ijkl}$, while for the second case they are for constant $\dot{\varepsilon}^d_{ij}$. As $\dot{\varepsilon}^d_{ij} = \dot{C}_{ijkl}\sigma_{kl}$, one can consider the maximization with constant degrading strain as a maximization problem on a reduced domain in which each point represents a whole family of points of the more general domain of compliances. Therefore, the maximum obtained when using associativity in strain space does not represent an absolute maximum of the degrading dissipation. To obtain an absolute maximum, one must complement the process with a maximization within the domain of all possible degradation models leading to the same flow rule. The results of this two-level maximization process is represented by the associated compliance rule (42b).

4.5. *Compliance rule as the gradient of a degradation potential* Q′

When formulated in terms of the second-order degradation rule m_{ij} [Section 3.2, eqns (18)–(21)], the present theory is formally identical to classical elastoplasticity. Therefore, the same assumption can be made that m_{ij} is the gradient of an elastic degradation potential Q in stress space. Now, after the thermodynamic force $-\mathbf{Y}$ has been introduced in Section 4.1, it is possible to go one step further and assume that $\mathbf{M}$ also derives from the gradients of a generalized degradation potential Q' in the $-\mathbf{Y}$ space. The corresponding expressions for $\mathbf{m}$ and $\mathbf{M}$ are

$$m_{ij} = \frac{\partial Q}{\partial\sigma_{ij}} \quad \text{and} \quad M_{ijkl} = \frac{\partial Q'}{\partial(-Y_{ijkl})}. \tag{43a, b}$$

Note that in general Q' does not need to be the same as Q, although both potentials are not independent since the resulting degradation rules $\mathbf{m}$ and $\mathbf{M}$ must always satisfy expression (25). As one might expected, the particular case $Q' = Q$ automatically satisfies (25) [this conclusion is straight forward if one replaces the right-hand side of (43a) by chain differentiation with intermediate variables $-Y_{pqrs}$, as done in eqns (37)–(38)].

With Q defined, one recovers the classical definition of associativity in plasticity with $Q = F$, although special cases must be considered, according to the discussion in the previous section. If one assumes $Q = F$ but $Q' \neq Q$, or if F is not a function of $-\mathbf{Y}$ (and therefore $\mathbf{N}$ is not defined), the formulation will be associated at the strain level but not at the compliance level. If, on the contrary, $\mathbf{N}$ exists and $Q' = Q$, the condition $Q = F$ implies that the formulation is fully associated at both strain and compliance levels.

5. STRAIN-BASED ELASTIC-DEGRADING FORMULATIONS

The concepts and developments presented in Sections 3 and 4 for the formulation of elastic-degrading materials with stress-based loading functions can be repeated in a dual way in strain space with a loading function $F[\boldsymbol{\varepsilon}, \bar{\mathbf{p}}]$. In this case, the rate equations analogous to eqns (18)–(20) are

$$\dot{\varepsilon}_{ij} = C_{ijkl}(\dot{\sigma}_{kl} - \dot{\sigma}^{\mathrm{d}}_{kl}) \quad \text{with} \quad \dot{\sigma}^{\mathrm{d}}_{ij} = -E_{ijkl}\dot{\varepsilon}^{\mathrm{d}}_{kl} \tag{44a, b}$$

$$\dot{\sigma}^{\mathrm{d}}_{kl} = \dot{\lambda}\bar{m}_{kl} \tag{45}$$

$$\dot{F} = \bar{n}_{ij}\dot{\varepsilon}_{ij} - \bar{H}\dot{\lambda} = 0 \quad \text{with} \quad \bar{n}_{ij} = \left.\frac{\partial F}{\partial \varepsilon_{ij}}\right|_{\lambda} \quad \text{and} \quad \bar{H} = -\left.\frac{\partial F}{\partial \lambda}\right|_{\varepsilon} = -\left.\frac{\partial F}{\partial \bar{p}_i}\right|_{\varepsilon}\frac{\partial \bar{p}_i}{\partial \sigma^{\mathrm{d}}_{kl}}m_{kl}. \tag{46a, b, c}$$

As before, there are two ways of combining these equations to obtain either the tangential compliance or the tangential stiffness. Due to the advantage of controlling also the softening regime, only the expression for the multiplier $\dot{\lambda}$ in terms of the strain rate and the tangential stiffness tensor will be considered here. The expressions are identical to the ones obtained for strain-based plasticity (13), provided the initial stiffness E^0_{ijkl} is replaced with the secant stiffness E_{ijkl}.

Analogously to what was done in Section 2.2, each strain-based degrading formulation can be related to a stress-based counterpart which is equivalent. The parameters $\bar{m}_{ij}$, $\bar{n}_{ij}$ and $\bar{H}$ can be related to their stress-based counterparts m_{ij}, n_{ij} and H by

$$\bar{m}_{ij} = -E_{ijkl}m_{kl} \quad \text{or} \quad m_{ij} = -C_{ijkl}\bar{m}_{kl} \tag{47a, b}$$

$$\bar{n}_{ij} = E_{ijkl}n_{kl} \quad \text{or} \quad n_{ij} = C_{ijkl}\bar{n}_{kl} \tag{47c, d}$$

$$\bar{H} = H + n_{ij}E_{ijkl}m_{kl} \quad \text{or} \quad H = \bar{H} + \bar{n}_{ij}C_{ijkl}\bar{m}_{kl} \tag{47e, f}$$

which are similar to eqns (14a–f), but involve the secant stiffness E_{ijkl} and secant compliance C_{ijkl} instead of the initial moduli tensors E^0_{ijkl} and C^0_{ijkl}.

In a dual approach to the stress-based formulation in Section 3.3, a degradation rule $\bar{M}_{ijkl}$ is assumed for the evolution of the secant stiffness, and the relation between $\bar{M}_{ijkl}$ and $\bar{m}_{ij}$ is obtained

$$\dot{E}_{ijkl} = \dot{\lambda}\bar{M}_{ijkl}; \quad \bar{m}_{ij} = \bar{M}_{ijkl}\varepsilon_{kl}. \tag{48a, b}$$

The final set of equations for the strain-based degrading material, dual to (26) and (27), include the tangential stiffness tensor and the evolution law for the secant stiffness

$$E^{\mathrm{t}}_{ijkl} = E_{ijkl} + \frac{1}{\bar{H}}\bar{M}_{ijpq}\varepsilon_{pq}\bar{n}_{kl} \quad \text{and} \quad \dot{E}_{ijkl} = \bar{M}_{ijkl}\frac{\bar{n}_{ab}\dot{\varepsilon}_{ab}}{\bar{H}}. \tag{49a, b}$$

The expressions for the degrading dissipation $\dot{d}$ in terms of the rate of stiffness $\dot{E}_{ijkl}$ and the expression for the corresponding generalized forces $-\bar{Y}_{ijkl}$, were already given in Section 4.1 by eqns (30a), (31a) and (32a). From (30a), the restriction that the rate of degrading dissipation must be positive translates in this case to $\dot{d} = -\varepsilon_{ij}\varepsilon_{kl}\bar{M}_{ijkl}\dot{\lambda}/2 \geqslant 0$ which, since $\dot{\lambda} \geqslant 0$, means that $\bar{M}_{ijkl}$ must be negative definite (opposite to the restriction on M_{ijkl}, which had to be positive). Using (35a) and (44b), $\dot{d}$ can be also expressed in terms of the degrading stress rate $\dot{\sigma}^{\mathrm{d}}$ and its conjugate thermodynamic force $-\bar{y}_{ij}$, as

$$\dot{d} = -\tfrac{1}{2}\varepsilon_{ij}\dot{\sigma}^{\mathrm{d}}_{ij} \quad \text{or} \quad \dot{d} = (-\bar{y}_{ij})\dot{\sigma}^{\mathrm{d}}_{ij}; \quad -\bar{y}_{ij} = -\tfrac{1}{2}\varepsilon_{ij}. \tag{50a, b, c}$$

The dual counterpart of N_{ijkl} and its relation to n_{ij} can be obtained as

$$\bar{N}_{ijkl} = \left.\frac{\partial F}{\partial(-\bar{Y}_{ijkl})}\right|_{\lambda} \quad \text{and} \quad \bar{n}_{ij} = -\bar{N}_{ijkl}\varepsilon_{kl}. \tag{51a, b}$$

Associativity in the stiffness space is obtained when $\bar{M}_{ijkl}$ is parallel to $\bar{N}_{ijkl}$. A similar argument to the one in Section 4.4 leads to the conclusion that associativity in the stiffness space implies associativity in the strain space, but not the opposite.

The degradation rule for stiffness $\bar{M}_{ijkl}$ in the strain-based formulation can be related to its counterpart for compliance M_{ijkl} used in the stress-based formulation, by introducing expressions (24) and (48a), into either of eqns (17). $\bar{N}_{ijkl}$ can also be related to N_{ijkl}:

$$\bar{M}_{ijkl} = -E_{ijpq}M_{pqrs}E_{rskl} \quad \text{or} \quad M_{ijkl} = -C_{ijpq}\bar{M}_{pqrs}C_{rskl} \tag{52a, b}$$

$$\bar{N}_{ijkl} = -E_{ijpq}N_{pqrs}E_{rskl} \quad \text{or} \quad N_{ijkl} = -C_{ijpq}\bar{N}_{pqrs}C_{rskl}. \tag{52c, d}$$

Finally, the relation between $\bar{H}$ and H was stated in (47e, f), where m_{ij}, n_{ij}, $\bar{m}_{ij}$ and $\bar{n}_{ij}$ can be expressed in terms of M_{ijkl}, N_{ijkl}, $\bar{M}_{ijkl}$ and $\bar{N}_{ijkl}$ according to (25), (38), (48b) and (51b) respectively.

The main relations for the stress-based as well as strain-based formulations of elastic-degrading models are summarized in Table 1 of the Appendix, where symbolic notation has been used for compactness.

In those tables the duality between the gradients of F ($\mathbf{n}$, $\mathbf{N}$, $\bar{\mathbf{n}}$ and $\bar{\mathbf{N}}$) and the flow rules ($\mathbf{m}$, $\mathbf{M}$, $\bar{\mathbf{m}}$ and $\bar{\mathbf{M}}$) is maintained except for some differences in signs. This is because the gradients in the compliance and stiffness spaces, $\mathbf{N}$ and $\bar{\mathbf{N}}$, are taken as derivatives with respect to thermodynamic forces $-\mathbf{Y}$ and $-\bar{\mathbf{Y}}$, while the gradients in strain and stress spaces are taken with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, which are not the actual thermodynamic forces in those spaces, $\mathbf{y}$ and $\bar{\mathbf{y}}$. These forces not only differ from $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ by a factor of 1/2, but also by a minus sign in the case of $\bar{\mathbf{y}}$ (in the case of elastoplasticity, Section 2, there is only change of minus sign without the factor of 1/2). If the gradient in stress space $\bar{\mathbf{n}}$ had been taken with respect to $-\boldsymbol{\varepsilon}$ (and $\mathbf{n}$ the same as now) this difference in signs would disappear and full duality would be recovered between the gradients (ns) and the degradation rules (ms). This change, however, would have generated additional changes that would disrupt the duality between the basic plasticity-like equations of the stress- and strain-based formulations in Sections 2 and 5. In particular, $\bar{\mathbf{n}}$ would then denote the *inward* normal to F in strain space (instead of the outward normal), and some signs would change in eqns (11a), (12a, b), (13a, b) and (14c, d and f) for strain-based elastoplasticity, and (46a, b), (47c, d and f), (49a, b), and (51b) for the analogous degrading formulation. The conclusion is that in the search for simplicity and duality in the theory, we should accept an intrinsic asymmetry that cannot be avoided and that will show up in the equations in one way or the other. This paper as much as possible has followed traditional plasticity conventions (e.g. $\bar{\mathbf{n}}$ = outward normal to F), and has concentrated the differences in the new formulation of unified theory of elastic degradation.

6. INTRODUCTION OF DAMAGE VARIABLES: GENERAL IMPLICATIONS FOR ED MODELS AND ASSOCIATIVITY IN THE DAMAGE SPACE

In the previous Sections 3–5, the stiffness degradation was defined directly in terms of the evolution of the secant compliance or stiffness tensors. That is, of course, the most general way of approaching the problem of elastic stiffness degradiation, but usually not the most simple and most effective. It is reasonable to assume that there exists a much more reduced set of parameters (far less than the 21 independent components of C_{ijkl} or E_{ijkl}), which characterize the state of degradation or *damage* achieved by the material at any state of the loading process. Let us call this reduced set of descriptors *the set of damage variables*, and designate them by the symbol $\mathscr{D}_*$. In practice, $\mathscr{D}_*$ can represent a scalar, a vector, a second-order tensor, or even a fourth-order tensor. There are, however, some interesting theoretical implications of assuming a set of damage variables which are independent of their specific nature and physical meaning. These general theoretical implications are discussed in this section. A more detailed review of the various types of damage variable and their particular implications on the resulting model is presented in a sequel paper.

6.1. *Stress- (or compliance-) based formulation*

In stress-based formulations of elastic-degrading materials, the degradation process is modeled by means of some rules which describe the progressive "increase" of the secant compliance tensor C_{ijkl} (Sections 3 and 4). The introduction of a set of damage variables $\mathscr{D}_*$ that fully characterize the state of degradation may be expressed as

$$C_{ijkl} = C_{ijkl}[C^0_{pqrs}, \mathscr{D}_*], \tag{53}$$

i.e. the components of the secant compliance tensor are given by a set of known, continuous and differentiable functions of the initial compliance C^0_{ijkl} and the damage variables $\mathscr{D}_*$. Independently of the (scalar-valued, vector or tensor) character of $\mathscr{D}_*$, one can always write after (53)

$$\dot{C}_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\dot{\mathscr{D}}_* \tag{54}$$

where the partial derivatives are also obtained by differentiation of (53), and repetition of the special subindex "$*$" implies summation over all indices represented by this symbol. A flow rule for damage can now be written as

$$\mathscr{D}_* = \dot{\lambda}\mathscr{M}_* \tag{55}$$

in which, similar to the flow rule for degrading strains (19) or the compliance rule (24), $\dot{\lambda}$ is a (scalar) *damage multiplier* defining the magnitude, and $\mathscr{M}_*$ defines the direction of the rate of change of the damage variables in the *damage space* ($\mathscr{M}_*$ has obviously the same scalar, vectorial or tensorial character, and dimensions as $\mathscr{D}_*$). By substituting (24) and (55) in (54), and by eliminating the scalar multiplier (by conveniently rescaling the flow rule), one obtains

$$M_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\mathscr{M}_*. \tag{56}$$

Since the partial derivatives are known functions, eqn (56) means that once the damage rule $\mathscr{M}_*$ is established, the compliance rule M_{ijkl} follows automatically. With (56), the final set of equations for the evolution of the elastic-damage model can be immediately obtained by replacing this expression into (26) and (27).

In order to discuss the concept of associativity, the normal to the loading surface in the damage space, $\mathscr{N}_*$ (similar to the normal in the strain space n_{ij}, or the normal in the compliance space N_{ijkl}) has to be defined. To do that, one resorts again to the thermodynamic concepts of mechanical free energy, w given by (28a) or (28b). The expression for the degrading dissipation given in (31b) can be rewritten by replacing $\dot{C}_{ijkl}$ by its expression (54) as

$$\dot{d} = (-Y_{ijkl})\frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\dot{\mathscr{D}}_*; \quad \text{or} \quad \dot{d} = (-\mathscr{Y}_*)\dot{\mathscr{D}}_* \quad \text{with} \quad -\mathscr{Y}_* = (-Y_{ijkl})\frac{\partial C_{ijkl}}{\partial \mathscr{D}_*} \tag{57a, b, c}$$

where $-\mathscr{Y}_*$ denotes the thermodynamic force conjugate to the damage variable $\mathscr{D}_*$, which must have the same (scalar, vectorial or tensorial) character and dimensions. Note that the previous definition of the thermodynamic force $-\mathscr{Y}_*$ is equivalent to its definition as the gradient of the mechanical free energy

$$\mathscr{Y}_* = \left.\frac{\partial w}{\partial \mathscr{D}_*}\right|_\varepsilon . \tag{58}$$

Assuming that the loading function F can be expressed in terms of the thermodynamic force $-\mathscr{Y}_*$ [which is given by (57c) with $-Y_{ijkl} = \sigma_{ij}\sigma_{kl}/2$], the normal to the loading function in the damage space can be defined as

$$\mathscr{N}_* = \left.\frac{\partial F}{\partial(-\mathscr{Y}_*)}\right|_\lambda \tag{59}$$

where $\mathscr{N}_*$ has the same character and dimensions as $\mathscr{D}_*$ and $\mathscr{Y}_*$. An elastic-damage formulation may be called *associated in the damage space* whenever $\mathscr{M}_*$ is proportional to $\mathscr{N}_*$. This is a third type of associativity for ED models, to be added to the two levels of associativity in the compliance space and strain space defined before. In the case that F cannot be expressed as a function of $-\mathscr{Y}_*$, then $\mathscr{N}_*$ is not defined and the concept of associativity at the damage level cannot be considered, i.e. the model is automatically non-associated at this level.

Associativity in the damage space has implications which are analogous to those at the compliance or strain levels. The relation between damage rate and the rate of the corresponding thermodynamic force is defined as

$$\dot{\mathscr{D}}_* = \mathscr{A}_{*\circ}(-\dot{\mathscr{Y}}_\circ) \quad \text{with} \quad \mathscr{A}_{*\circ} = \frac{1}{H}\mathscr{M}_*\mathscr{N}_\circ \tag{60a, b}$$

in which the subscript "$\circ$" indicates a set of indices similar to the one represented by subindex "$*$". The tensor $\mathscr{A}_{*\circ}$ (60b) exhibits major symmetry when $\mathscr{M}_*$ and $\mathscr{N}_*$ are proportional. Also, the rate of dissipation (58a) is maximized by using an associated damage rule (this can be obtained by considering stationary values of the Lagrangian $\mathscr{L}^d = \mathscr{Y}_*\dot{\mathscr{D}}_* + \dot{\lambda}F$ with respect to the variation of $\mathscr{Y}_*$).

Associativity in the damage space is related to associativity in the compliance space, and therefore also to associativity in the strain space. First, note that the existence of $\mathscr{N}_*$ implies that of N_{ijkl} because the former means that F is a function of $-\mathscr{Y}_*$, and given its expression (57c), this means that F is also a function of $-Y_{ijkl}$. Now, (56) provides the relation between $\mathscr{M}_*$ and M_{ijkl}. To obtain a similar relation between $\mathscr{N}_*$ and N_{ijkl}, (36) must be rewritten in terms of the intermediate variable $\mathscr{Y}_*$:

$$N_{ijkl} = \frac{\partial F}{\partial(-\mathscr{Y}_*)}\frac{\partial(-\mathscr{Y}_*)}{\partial(-Y_{ijkl})} = \mathscr{N}_*\frac{\partial(-\mathscr{Y}_*)}{\partial(-Y_{ijkl})}. \tag{61}$$

The term $\partial(-\mathscr{Y}_*)/\partial(-Y_{ijkl})$ can be related to $\partial C_{ijkl}/\partial\mathscr{D}_*$. This is achieved by partial differentiation of (57c) with respect to $-Y_{ijkl}$, for constant damage $\mathscr{D}_*$ [and therefore, according to eqn (54), also constant compliance and its derivatives]. As the result, one can write

$$\frac{\partial(-\mathscr{Y}_*)}{\partial(-Y_{ijkl})} = \frac{\partial C_{ijkl}}{\partial\mathscr{D}_*} \tag{62}$$

which, introduced into (61) and after reordering terms, yields

$$N_{ijkl} = \frac{\partial C_{ijkl}}{\partial\mathscr{D}_*}\mathscr{N}_*. \tag{63}$$

This relation between $\mathscr{N}_*$ and N_{ijkl} is the same as the one between $\mathscr{M}_*$ and M_{ijkl} (56). This means that whenever $\mathscr{N}_*$ exists and $\mathscr{M}_*$ is parallel to it, the corresponding M_{ijkl} is parallel to the corresponding N_{ijkl} and, therefore, associativity in the damage space necessarily implies associativity in the compliance space.

Similarly to the degradation rule M_{ijkl}, the damage rule $\mathscr{M}_*$ can be assumed to emanate from a potential Q'' as

$$\mathscr{M}_* = \frac{\partial Q''}{\partial(-\mathscr{Y}_*)}. \tag{64}$$

Note that Q'' does not need to be the same as Q' or Q in Section 4.5, although these three potentials are related since their derivatives (64) and (43) must satisfy expressions (56) and (25). An obvious particular case, that automatically satisfies those requirements, is when $Q'' = Q' = Q$. Then, associativity at the damage level, and therefore full associativity, can be stated in similar terms as in classical plasticity, i.e. when $Q = F$.

6.2. *Strain- (or stiffness-) based formulation*

The dual approach of strain-based formulations in Section 5 represents the degradation process by means of the progressive "reduction" of the secant stiffness tensor E_{ijkl}. In these formulations, the introduction of a set of damage variables $\bar{\mathscr{D}}_*$ that fully characterize the state of degradation, can be expressed as

$$E_{ijkl} = E_{ijkl}[E^0_{pqrs}, \bar{\mathscr{D}}_*] \quad \text{and} \quad \dot{E}_{ijkl} = \frac{\partial E_{ijkl}}{\partial \bar{\mathscr{D}}_*} \dot{\bar{\mathscr{D}}}_*. \tag{65a, b}$$

Note that the damage variable $\bar{\mathscr{D}}_*$ is not the same as $\mathscr{D}_*$ used for the stress-based formulation. They both represent the same physical phenomenon, but for practical reasons, different damage variables may be used in stress-based or strain-based formulations (although normally they are related and could for instance be inverse to each other). Again, a damage rule $\bar{\mathscr{M}}_*$ is assumed for $\bar{\mathscr{D}}_*$ and is related to the stiffness rule $\bar{M}_{ijkl}$

$$\dot{\bar{\mathscr{D}}}_* = \dot{\lambda} \bar{\mathscr{M}}_* \quad \text{and} \quad \bar{M}_{ijkl} = \frac{\partial E_{ijkl}}{\partial \bar{\mathscr{D}}_*} \bar{\mathscr{M}}_*. \tag{66a, b}$$

The final set of equations for the evolution of a general strain-based ED model is readily obtained by substituting $\bar{M}_{ijkl}$ from (66b) into eqns (49a, b).

Similar considerations as in the previous section lead to the dissipation rate and the thermodynamic force $-\bar{\mathscr{Y}}_*$ conjugate to $\bar{\mathscr{D}}_*$

$$\dot{d} = (-\bar{\mathscr{Y}}_*)\dot{\bar{\mathscr{D}}}_* \quad \text{with} \quad -\bar{\mathscr{Y}}_* = (-\bar{Y}_{ijkl}) \frac{\partial E_{ijkl}}{\partial \bar{\mathscr{D}}_*} \tag{67a, b}$$

or its equivalent definition in terms of the free energy

$$\bar{\mathscr{Y}}_* = \left. \frac{\partial w}{\partial \bar{\mathscr{D}}_*} \right|_\varepsilon. \tag{68}$$

The normal to the loading surface in the stiffness space $\bar{\mathscr{N}}_*$ and its relation to $\bar{N}_{ijkl}$ are given by

$$\bar{\mathscr{N}}_* = \left. \frac{\partial F}{\partial(-\bar{\mathscr{Y}}_*)} \right|_\lambda \quad \text{and} \quad \bar{N}_{ijkl} = \frac{\partial E_{ijkl}}{\partial \bar{\mathscr{D}}_*} \bar{\mathscr{N}}_*. \tag{69a, b}$$

The model is associated in the damage space when $\bar{\mathscr{N}}_*$ is proportional to $\bar{\mathscr{M}}_*$. The relations of these damage quantities to their counterparts in the stiffness space $\bar{M}_{ijkl}$ (66b) and $\bar{N}_{ijkl}$ (69b), show that associativity in the damage space necessarily implies associativity in the stiffness space and therefore also associativity in the stress space. The discussion in the last section about the consequences of associativity at the damage level is also valid here, as well as the possibility of considering $\bar{\mathscr{M}}_*$ emanating from a potential $\bar{Q}''$, etc.

Similarly to what happened with flow rules for strain and stress (14), or compliance and stiffness (52), the flow rules for both types of damage, $\mathscr{M}_*$ and $\bar{\mathscr{M}}_*$, are related to each other. Replacing $\bar{M}_{ijkl}$ and M_{ijkl} by their expressions (56) and (66b), into (52a) or (52b), and after some mathematical manipulation, one obtains

$$\bar{\mathscr{M}}_* = -\left[\frac{\partial E_{abcd}}{\partial \bar{\mathscr{D}}_*}\frac{\partial E_{abcd}}{\partial \bar{\mathscr{D}}_\circ}\right]^{-1}\frac{\partial E_{ijkl}}{\partial \bar{\mathscr{D}}_\circ}E_{klpq}\frac{\partial C_{pqrs}}{\partial \mathscr{D}_\triangleleft}E_{rsij}\mathscr{M}_\triangleleft \tag{70a}$$

$$\mathscr{M}_* = -\left[\frac{\partial C_{abcd}}{\partial \mathscr{D}_*}\frac{\partial C_{abcd}}{\partial \mathscr{D}_\circ}\right]^{-1}\frac{\partial C_{ijkl}}{\partial \mathscr{D}_\circ}C_{klpq}\frac{\partial E_{pqrs}}{\partial \bar{\mathscr{D}}_\triangleleft}C_{rsij}\bar{\mathscr{M}}_\triangleleft \tag{70b}$$

where the symbols "$\circ$" and "$\triangleleft$" are used to indicate different sets of indices similar to "$*$", and where repeated imply summation over all indices in the set. Similar expressions can be derived to relate $\bar{\mathscr{N}}_*$ and $\mathscr{N}_*$. Equations (47e, f) relate $\bar{H}$ to H where, if desired, m_{ij}, n_{ij}, $\bar{m}_{ij}$ and $\bar{n}_{ij}$ can be replaced by M_{ijkl} (25), N_{ijkl} (38), $\bar{M}_{ijkl}$ (48b) and $\bar{N}_{ijkl}$ (51b), and these can be replaced by $\mathscr{M}_*$ (56), $\mathscr{N}_*$ (63), $\bar{\mathscr{M}}_*$ (66b) and $\bar{\mathscr{N}}_*$ (69b).

The main equations for elastic-damage models are summarized in compact notation in Tables 2 and 3 of the Appendix for the compliance-based and stiffness-based formulations respectively.

7. SCALAR DAMAGE MODELS FROM THE LITERATURE AS EXAMPLES OF THE THEORY

The unified theory presented in previous sections encompasses many of the continuum damage models which have been proposed in the recent literature. Those models are usually introduced from a thermodynamic potential; most of them start from what seem to be different assumptions, and therefore they look like entirely different models, or at least the underlying common framework is not evident.

In this section, some of the most relevant existing scalar damage models are revisited. In the following paragraphs, they are reformulated from their basic assumptions in the context of the proposed unified theory, omitting any additional features such as combination of damage and plasticity, positive–negative projection operators, etc. Using the unified theory, the derivations are straightforward and easy to understand when we start from a loading surface and a flow rule rather than a thermodynamic potential. Moreover, explicit expressions of the tangent stiffness or compliance operators are obtained in a more direct and systematic way. The derivation of the various models in the context of a single general theory makes it possible to compare the similarities and differences between them, and to extract a common underlying structure of the tangent operator for all associated scalar damage models of the traditional $(1-D)$ type. The general format of the tangent operator is interesting not only for demonstrating that there exists a common framework, but also for subsequent analysis of elastic material degradation with regard to localization and other failure conditions (Rizzi *et al.*, 1993).

7.1. *Mazars and Lemaitre* (1984)

Mazars and Lemaitre (1984) proposed an isotropic scalar damage model based on a thermodynamic potential. In that case a single scalar damage variable defines the relation between nominal stresses σ_{ij} and effective stresses σ^0_{ij}, in the spirit of the original damage interpretation by Kachanov (1958). The same relation can be applied to define the relation between initial and current secant stiffness

$$\sigma_{ij} = (1-D)\sigma^0_{ij}; \quad E_{ijkl} = (1-D)E^0_{ijkl}. \tag{71a, b}$$

In the terminology used in this paper, Mazars–Lemaitre's model is a strain-based damage formulation. Equation (71b) is the specific form of the general expression (65a) with the scalar damage variable $\bar{\mathscr{D}}_* = D$. The loading function is defined in the strain space as a function of the second invariant of ε_{ij}

$$F[\bar{\varepsilon}, D] = \bar{\varepsilon} - \bar{r}[D] \quad \text{where} \quad \bar{\varepsilon} = \sqrt{\varepsilon_{ij}\varepsilon_{ij}}. \tag{72a, b}$$

With definitions (72a, b), the model is non-associated. In fact, after differentiation of (71b), one obtains

$$\dot{E}_{ijkl} = -\dot{D}E^0_{ijkl}. \tag{73}$$

By comparison with eqns (65b) and (66a), one can identify

$$\frac{\partial E_{ijkl}}{\partial D} = -E^0_{ijkl}; \quad \bar{\mathscr{M}} = 1; \quad \dot{\lambda} = \dot{D}. \tag{74a, b, c}$$

The assumption $\bar{\mathscr{M}} = 1$ is the same for all strain-based scalar damage models, since this suffices to define a "direction" in the (one-dimensional) damage space. Further, by using (66b) and (48b), one can develop the corresponding flow rules in both stiffness and stress spaces of an elastic-degrading material,

$$\bar{M}_{ijkl} = -E^0_{ijkl} \quad \text{and} \quad \bar{m}_{ij} = -E^0_{ijkl}\varepsilon_{kl} = -\sigma^0_{ij}. \tag{75a, b}$$

From (72a), by using (46b), (46c) and (74c), the expressions of the normal to the loading surface and $\bar{H}$ are

$$\bar{n}_{kl} = \frac{\varepsilon_{kl}}{\bar{\varepsilon}} \quad \text{and} \quad \bar{H} = \frac{\partial \bar{r}}{\partial D}. \tag{76a, b}$$

All these results can be substituted into (49a), to develop the following tangential operator

$$E^t_{ijkl} = (1-D)E^0_{ijkl} - \frac{1}{\bar{\varepsilon}\dfrac{\partial \bar{r}}{\partial D}}\sigma^0_{ij}\varepsilon_{kl}. \tag{77}$$

This expression was in fact not presented originally by Mazars and Lemaitre (1984), but was developed later as eqn (18) by Simo and Ju (1987) [where the coefficient of the second term in the right-hand side is written as $H/\bar{\tau}$, with $H = 1/(\partial\bar{r}/\partial D)$ and $\bar{\tau} = \bar{\varepsilon}$]. In the way the loading function F is defined (72), its derivatives with respect to strain, $\bar{n}_{ij}$ (76a), are not proportional to the flow rule for strains $\bar{m}_{ij}$ (75b), and therefore the model is non-associated at the strain level, and the tangent operator (77) is not symmetric. Non-associativeness at the strain level also implies that the model is non-associated at the stiffness and damage levels. In fact, one can verify that independently at the stiffness level, by computing $\bar{N}_{ijkl}$ (51a) as $\bar{N}_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})/2\bar{\varepsilon}$, and realizing that it is not proportional to the $\bar{M}_{ijkl}$ in (75a). At the damage level, associativity cannot even be considered because F (72) cannot be expressed as a function of the thermodynamic force $-\mathscr{Y}$ that, given (57c), (74a) and (32a), is defined in this case as the strain-based undamaged elastic energy:

$$-\mathscr{Y} = \bar{w}^0 ; \quad \bar{w}^0 = \tfrac{1}{2}\varepsilon_{ij} E^0_{ijkl} \varepsilon_{kl}. \tag{78a, b}$$

7.2. *Simo and Ju* (1987)

Simo and Ju (1987) proposed a scalar isotropic damage model based on a certain definition of the free energy potential. The model was presented in both stress- and strain-based forms. The authors considered further extensions to anisotropic damage, with the introduction of positive projection operators for tensile damage. Keeping in mind the purpose of the present discussion, only the scalar elastic damage model will be addressed in this section.

(a) *Strain-based model.* The definition of the model is very similar to the model in the previous section [scalar damage, strain-based, eqns (71a, b) and (73)–(75)], except for the definition of the loading function, which in this case reads

$$F[\bar{\tau}, D] = \bar{\tau} - \bar{r}'[D] \tag{79}$$

where the scalar quantity $\bar{\tau}$ is a function of the strain-based undamaged elastic energy $\bar{w}^0$ (78b)

$$\bar{\tau} = \sqrt{2\bar{w}^0}. \tag{80}$$

With this definition of the loading function, one can obtain

$$\bar{n}_{kl} = \frac{\sigma^0_{kl}}{\bar{\tau}} \quad \text{and} \quad \bar{H} = \frac{\partial \bar{r}'}{\partial D}. \tag{81a, b}$$

The tangent operator is evaluated by substituting E_{ijkl} (71b), $\bar{M}_{ijkl}$ (75a), $\bar{n}_{kl}$ (81a) and $\bar{H}$ (81b) into the general form (49a):

$$E^{\mathrm{t}}_{ijkl} = (1-D)E^0_{ijkl} - \frac{1}{\bar{\tau}\dfrac{\partial \bar{r}'}{\partial D}} \sigma^0_{ij}\sigma^0_{kl}. \tag{82}$$

This is the same as eqn (17) of Simo and Ju (1987) [where the coefficient of the second term on the right-hand side was written as $H/\bar{\tau}$, with $H = 1/(\partial \bar{r}'/\partial D)$]. In contrast to Mazars–Lemaitre, this model is fully associated at all levels (damage, stiffness and strain), since by differentiation of the loading function (79) with (69a) and (78), and application of (69b) and (51b), one can obtain

$$\bar{\mathscr{N}} = \frac{1}{\bar{\tau}}; \quad \bar{N}_{ijkl} = \frac{1}{\bar{\tau}}(-E^0_{ijkl}); \quad \bar{n}_{ij} = \frac{1}{\bar{\tau}}\sigma^0_{ij}. \tag{83a, b, c}$$

All these gradients turn out to be proportional to their counterparts

$$\bar{\mathscr{M}} = 1; \quad \bar{M}_{ijkl} = -E^0_{ijkl}; \quad \bar{m}_{ij} = -\sigma^0_{ij}. \tag{84a, b, c}$$

As a consequence of the associativity at the strain level, the tangent operator (82) exhibits major symmetry. Since for all scalar damage models $\bar{\mathscr{N}}$ and $\bar{\mathscr{M}}$ are scalar quantities, in the case that $\bar{\mathscr{N}}$ exists, they will be always parallel to each other. Therefore these scalar damage models will be fully associated (associativity at the damage level implies associativity at the stiffness and strain level; see Section 6.1). The only condition for a strain-based scalar damage model of the $(1-D)$ type to be fully associated is therefore the existence of a loading function of the kind $F = f[-\mathscr{Y}] - k$, where $-\mathscr{Y} = \bar{w}^0$ according to (78a, b).

(b) *Stress-based model.* In the dual stress-based formulation, the same expression for the secant stiffness (71b) is assumed, although it is more conveniently expressed in terms of the compliance as

$$C_{ijkl} = \bar{D} C^0_{ijkl}. \tag{85}$$

This is a specific form of the general expression (53) with scalar damage variable $\mathscr{D}_* = \bar{D}$. Note that this equation is fully equivalent to its stiffness-based counterpart (71b) with $\bar{D} = 1/(1-D)$. By differentiation of (85) one can identify the various terms of the theory (54)–(55)

$$\dot{C}_{ijkl} = \dot{\bar{D}} C^0_{ijkl}; \quad \dot{\lambda} = \dot{\bar{D}}; \quad \frac{\partial C_{ijkl}}{\partial \bar{D}} = C^0_{ijkl}; \quad \mathscr{M} = 1. \tag{86a, b, c, d}$$

Using (56) and (25) one can obtain the corresponding flow rules for the compliance and strain

$$M_{ijkl} = C^0_{ijkl}; \quad m_{ij} = C^0_{ijkl}\sigma_{kl} = \varepsilon^0_{ij}. \tag{87a, b}$$

The loading function is defined in the stress space with a similar expression where now the scalar quantity τ is a function of the stress state related to the undamaged complementary energy norm of the stress tensor:

$$F[\tau, \bar{D}] = \tau - r[\bar{D}]; \quad \tau = \sqrt{2w^0}; \quad w^0 = \tfrac{1}{2}\sigma_{ij} C^0_{ijkl}\sigma_{kl}. \tag{88a, b, c}$$

The gradient of the loading function and the hardening parameter are directly obtained by differentiation of F with respect to σ_{ij} and $\bar{D}$ [which, according to eqn (86b), is the same as λ]

$$n_{kl} = \frac{\varepsilon^0_{kl}}{\tau} \quad \text{and} \quad H = \frac{\partial r}{\partial \bar{D}} \tag{89a, b}$$

where $\varepsilon^0_{kl} = C^0_{klpq}\sigma_{pq}$. Since $\mathbf{n}$ is proportional to $\mathbf{m}$, the model is associated at the strain level and the tangential stiffness tensor will be symmetric. The expression for the tangent stiffness can be obtained by direct substitution of the previous terms into (21b)

$$E^t_{ijkl} = (1-D)E^0_{ijkl} - \frac{(1-D)^2}{\tau(\partial r/\partial \bar{D} + (1-D)\tau)}\sigma_{ij}\sigma_{kl}. \tag{90}$$

This equation, however, was not given in the original paper. The authors defined E^t_{ijkl} as the inverse of the tangent compliance C^t_{ijkl} [eqn (67) of Simo and Ju (1987)], which is not a convenient format since the compliance operator is only defined in the hardening regime, while the stiffness does exist in both hardening and softening regimes. Due to this limitation, the expression C^t_{ijkl} was not detailed in Section 3, although it is completely analogous to its elastoplastic counterpart (8b) if the initial stiffness C^0_{ijkl} is replaced by the secant C_{ijkl}. By doing that, one obtains

$$C^t_{ijkl} = \frac{1}{(1-D)} C^0_{ijkl} + \frac{1}{\tau\, \partial r/\partial \bar{D}} \varepsilon^0_{ij}\varepsilon^0_{kl}. \tag{91}$$

This equation matches the expression of the original paper with the following changes in notation: $d_\sigma = 1/(1-D)$, $\partial\Lambda^0_{ij}/\partial\sigma_{ij} = \varepsilon^0_{ij}$, $\partial^2\Lambda^0_{ij}/\partial(\sigma_{ij}\sigma_{kl}) = C^0_{ijkl}$ and $H = 1/(\partial r/\partial \bar{D})$.

Finally, some remarks with respect to full associativity can also be made here. With the definition of the secant compliance [(85a) and (86a)], the thermodynamic force in the damage space (57c) takes the scalar form $-\mathscr{Y} = w^0$. Thus, the only condition for associativity in the damage space (and therefore full associativity) is that F can be expressed as a function of that quantity. Since this is the case for (88a, b, c), the model is not only associated at the strain level as stated in a previous paragraph, but also fully associated at all three levels (strain, compliance, damage).

7.3. *Ju* (1989)

In a more recent work, Ju (1989) reconsidered the previous model with Simo and formulated a new elastoplastic damage model with a unique loading function for both damage and plasticity. Following the thermodynamic approach to damage used by a number of authors, the model is formulated on the basis of the Helmholtz free energy function in strain space, the stresses are defined as its derivatives with respect to strain and the stiffness as its second derivatives, and the thermodynamic force is defined as the first derivative of the same potential with respect to the damage variables. In the context of the present theory, only the elastic-damage part of that model with a constant stiffness upon unloading is considered. Under these conditions, Ju's elastic strain is equal to the total strain, the free energy function coincides with our w given by (28a, b), and the thermodynamic force with $-\mathscr{Y}_*$ in Section 6.2.

In Ju's work, the formal presentation of the model is supplemented by the assumptions that there exists a scalar damage variable D, and that the free energy function is equal to the product of the factor $(1-D)$ times some undamaged energy function, Ψ^0, which remains unspecified throughout the paper. In our terminology, this assumption translates into $w = (1-D)\bar{w}^0$ with $\bar{w}^0$ = strain-based undamaged elastic energy (78b), and implies the traditional expressions of the effective stresses (71a), secant stiffness (71b) and thermodynamic force (78a) for scalar damage models of the $(1-D)$ type, similarly to the previous strain-based models by Mazars–Lemaitre and Simo–Ju.

In the restricted version of Ju's model being considered here, the only significant change as compared to the earlier formulation by Simo–Ju is the definition of the loading function, which now reads (in our terminology)

$$F = \bar{w}^0 - \bar{r}''[D] \tag{92}$$

with $\bar{w}^0$ from (78b). Thus the loading function is a direct function of the undamaged free energy instead of the square root of its double (80), and consequently $\bar{r}''(D)$ is also not the same as $\bar{r}'$. From the definition of F one can obtain the gradients

$$\bar{n}_{ij} = \sigma^0_{ij} \quad \text{and} \quad \bar{H} = \frac{\partial \bar{r}''}{\partial D} \tag{93a, b}$$

and the tangent stiffness operator

$$E^t_{ijkl} = (1-D)E^0_{ijkl} - \frac{1}{\partial \bar{r}''/\partial D}\sigma^0_{ij}\sigma^0_{kl} \tag{94}$$

which is the same as given by Ju [eqn (21) in that paper, with the notation $E^0_{ijkl} = \partial^2\Psi^0/\partial\varepsilon_{ij}\partial\varepsilon_{kl}$ and $\partial\bar{r}''/\partial D = 1/H$]. As it could be expected, this tangent operator (94) is noticeably the same as that of Simo and Ju (82), if $\partial\bar{r}''/\partial D = \bar{\tau}\,\partial\bar{r}'/\partial D$. This equivalence can be verified easily by imposing $F = 0$ in (79) and (92) and combining the resulting expressions with (80).

7.4. *Benallal* et al. (1989)

Benallal *et al.* (1989) presented a general constitutive description for materials with stiffness degradation. The paper first includes a general formulation that would correspond, in the terminology of this paper, to the strain-based damage formulation of Section 6, with a different notation and some additional restrictions. Similarly to other authors, they start defining a free energy potential that is a function of strains and damage (scalar, vector or tensor) $\Psi[\varepsilon_{ij}, \mathscr{D}_*]$, from which the stresses and secant stiffness can be defined as its first and second derivatives with respect to strain, and the thermodynamic forces $-\mathscr{Y}_*$ as its first derivatives with respect to damage (68). Note that in the context of the present theory, the free energy potential is given by the elastic free energy w (28a, b).

A loading function is established in terms of the damage forces

$$F[-\mathscr{Y}_*, \mathscr{D}_*] \leqslant 0. \tag{95}$$

Note that this function is given directly as a function of the damage forces $-\mathscr{Y}_*$, instead of stresses or strains, which according to a previous discussion ensures full associativity for scalar damage. Together with being exclusively strain-based, this is the second restriction of this general formulation. As pointed out earlier, certain loading functions can be expressed in terms of stresses or strains but not in terms of the damage force, as for instance is the case for the damage model by Mazars and Lemaitre in Section 7.1. This type of damage models, which are necessarily non-associated at damage level, is therefore not included in the general formulation by Benallal *et al.* (1989).

The evolution of the damage variables is expressed by the traditional flow expression (66a), with the damage rule equal to the gradient of a potential $\bar{Q}$, i.e.

$$\dot{\mathscr{D}}_* = \dot{\lambda}\frac{\partial \bar{Q}}{\partial \bar{\mathscr{Y}}_*}. \tag{96}$$

Note that this is a further restriction since one can think of damage rules that cannot be expressed as the gradient of a potential.

With these ingredients, the consistency condition is imposed, and an expression for the tangential operator E^{t}_{ijkl} is obtained [eqn (8) in their paper], that is essentially identical to (49a), with the following notation: $\bar{M}_{ijkl}\varepsilon_{kl} = \bar{m}_{ij} = -\Lambda_{ij*}\,\partial F/\partial A_*$, $\bar{n}_{ij} = -\Lambda_{*ij}\,\partial f/\partial A_*$ and $\bar{H} = h$. These equivalences imply the following changes in basic notation: $F = f$, $\bar{Q} = F$, $\mathscr{D}_* = \alpha_*$, $-\mathscr{Y}_* = A_*$, $\bar{\mathscr{M}}_* = \partial F/\partial A_*$, $\bar{\mathscr{N}}_* = df/\partial A_*$ and the definition of $\Lambda_{ij*} = \varepsilon_{kl}\,\partial E_{ijkl}/\partial \mathscr{D}_*$.

The traditional strain-based scalar damage model of the $(1-D)$ type is then outlined in the paper as an example of the general formulation. The free energy is assigned in the usual way $w = (1-D)\bar{w}^0$ with $\bar{w}^0 = E^0_{ijkl}\varepsilon_{ij}\varepsilon_{kl}/2$ (78b), which implies the usual expressions of the effective stresses (71a), secant stiffness (71b), flow rule for stiffness (75a), flow rule for stresses (75b) and damage force (78a), similarly to the models by Mazars–Lemaitre, Simo–Ju and Ju discussed in the previous subsections. The loading function is identical to Ju's model (92), and so are the expressions of $\bar{n}_{ij}$ (93a), $\bar{H}$ (93b) and the resulting tangential stiffness E^{t}_{ijkl} (94), which is given in the original paper as eqn (22) with the following notation: $E^{t}_{ijkl} = L_{ijkl}$, $E^0_{ijkl} = A_{ijkl}$, $\sigma^0_{ij} = E^0_{ijpq}\varepsilon_{pq}$ and $\bar{H} = M$.

7.5. *Neilsen and Schreyer* (1992)

Neilsen and Schreyer (1992) proposed recently a damage formulation that fits very well into the general theory of elastic degradation, since they assume that the state of damage is directly characterized by the secant stiffness E_{ijkl}. The formulation is not starting from thermodynamic assumptions, rather, it starts from the definition of the secant relation $\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$ (15a), and the general flow rule of the stiffness change (48a):

$$\dot{E}_{ijkl} = \dot{\lambda}\bar{M}_{ijkl}. \tag{97}$$

In their paper [eqn (24)] the notation $\dot{\lambda} = \dot{w}$ and $\bar{\mathbf{M}} = -\mathbf{R}$ is used. Assumption (97) makes the formulation strain- (or stiffness-) based.

A loading function is assumed of the type

$$F = \tfrac{1}{2}\sigma_{ij} P_{ijkl} \sigma_{kl} - g[E_{ijkl}] \tag{98}$$

where P_{ijkl} is a general symmetric, constant, positive-definite, fourth-order tensor and g is a function of the damage state. Note that the loading function is given in terms of stresses, while the formulation is strain-based, and therefore the gradients naturally needed ($\bar{\mathbf{N}}$ and/or $\bar{\mathbf{n}}$) are derivatives with respect to strain. This is, however, not a major difficulty since the dual formulation based on the same F and the same multiplier $\dot{\lambda}$ is equivalent and therefore will offer the same tangential stiffness. By doing that, one needs the gradients in the stress space, which are:

$$N_{ijkl} = \frac{\partial F}{\partial(\frac{1}{2}\sigma_{ij}\sigma_{kl})} = P_{ijkl}; \quad n_{ij} = \frac{\partial F}{\partial \sigma_{ij}} = N_{ijkl}\sigma_{kl} = P_{ijkl}\sigma_{kl}. \tag{99a, b}$$

From $\bar{\mathbf{M}}$ (97), one can use (48b) to calculate $\bar{\mathbf{m}}$, and from that (47b) to obtain $\mathbf{m}$, or alternatively (52b) to obtain $\mathbf{M}$ and then (25) to obtain $\mathbf{m}$

$$\bar{m}_{ij} = \bar{M}_{ijkl}\varepsilon_{kl}; \quad M_{ijkl} = -C_{ijpq}\bar{M}_{pqrs}C_{rskl}; \quad m_{ij} = -C_{ijpq}\bar{M}_{pqrs}\sigma_{rs}. \tag{100a, b, c}$$

With $\mathbf{n}$ and $\mathbf{m}$, the tangent stiffness (21b) can be written as

$$E^{t}_{ijkl} = E_{ijkl} - \frac{-\bar{M}_{ijpq}\varepsilon_{pq} n_{rs} E_{rskl}}{H - n_{ab}\bar{M}_{abcd}\varepsilon_{cd}} \tag{101}$$

where the hardening–softening function H is given by

$$H = -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial E_{ijkl}}\bar{M}_{ijkl} \quad \text{or} \quad H = \frac{\partial g}{\partial E_{ijkl}}\bar{M}_{ijkl} \tag{102a, b}$$

with all partial derivatives being evaluated at constant stress. Equations (101) and (102a) are the same expressions obtained by Neilsen and Schreyer [eqns (25)–(27) in their paper, with $\mathbf{E}^{t} = \mathbf{D}$, $\mathbf{E} = \mathbf{S}$, $\bar{\mathbf{M}} = -\mathbf{R}$ and $\mathbf{n} = \mathbf{f}$]. The condition for symmetry of the tangential operator is obviously that $\bar{M}_{ijpq}\varepsilon_{pq} = n_{rs}E_{rsij}$, and in the original paper this is said (though not proven) to happen when $\bar{M}_{ijkl} = -E_{ijpq}P_{pqrs}E_{rskl}$. In the context of this theory, however, this is a trivial result because with $\bar{\mathbf{N}}$ evaluated from (99a) and (52c), this implies that $\bar{\mathbf{M}} = \bar{\mathbf{N}}$.

A more specific formulation was subsequently given by Neilsen and Schreyer as an example of the general theory. It is the traditional stress-based scalar damage model of the $(1-D)$ type, where (similarly to Mazars–Lemaitre, Simo–Ju, Ju, and Benallal *et al.*) the secant stiffness is $E_{ijkl} = (1-D)E^{0}_{ijkl}$ (71b), and the degradation rule in terms of stiffness (75a) and the corresponding rule in terms of compliance (52b) are

$$\bar{M}_{ijkl} = -E^{0}_{ijkl}; \quad M_{ijkl} = -C_{ijpq}\bar{M}_{pqrs}C_{rskl} = \frac{1}{(1-D)^2}C^{0}_{ijkl}. \tag{103a, b}$$

The loading function is given by (98) with $\mathbf{P} = \mathbf{C}^{0}$. Note that in this way $\mathbf{N} = \mathbf{P}$ is proportional to $\mathbf{M}$ (103b), and F directly contains the stress-based undamaged elastic energy and therefore the model becomes fully associated. Also, the gradients can be immediately calculated from (99a, b):

$$F = \tfrac{1}{2}\sigma_{ij}C^{0}_{ijkl}\sigma_{kl} - g[D]; \quad N_{ijlk} = C^{0}_{ijkl}; \quad n_{ij} = C^{0}_{ijkl}\sigma_{kl} \tag{104a, b, c}$$

and, since now one has $\dot{\lambda} = \dot{D}$ the hardening function is

$$H = -\frac{\partial F}{\partial \lambda} = \frac{\partial g}{\partial D} \tag{105}$$

where the partial derivatives are evaluated at constant stress. The resulting tangential stiffness is obtained by replacing $\bar{M}$, $\mathbf{n}$ and H into (101) which, after some rearrangements, yields

$$E^{t}_{ijkl} = (1-D)E^{0}_{ijkl} - \frac{1}{\partial g/\partial D + \varepsilon_{pq}\sigma_{pq}}\sigma_{ij}\sigma_{kl}. \tag{106}$$

This expression matches exactly the one given in the original paper [eqn (35) with $\mathbf{E}^{t} = \mathbf{D}$, $\mathbf{E} = \mathbf{S}$ and $\partial g/\partial D = H$].

8. GENERAL FORMULATION OF ASSOCIATED SCALAR DAMAGE MODELS OF THE $(1-D)$ TYPE

In the previous section, it has been seen that all the associated scalar models reviewed lead to a similar form of the symmetric tangent operator, with a rank-one modification of the secant stiffness involving a dyadic product of two stress tensors [eqns (82), (90), (94) and (106)]. By using the unified theory proposed in this paper, one can in fact obtain a more general formulation for associated traditional scalar damage models of the $(1-D)$ type, that includes the previous formulations (both stress- and strain-based) as particular cases, and that includes many other possible models of a similar format of the tangent operator, i.e.

$$E^{t}_{ijkl} = (1-D)E^{0}_{ijkl} - \frac{1}{\bar{h}(\sigma_{pq}, \varepsilon_{rs}, D)}\sigma_{ij}\sigma_{kl}. \tag{107}$$

In order to do that, consider again the classical assumptions of secant stiffness and the corresponding degradation rules for stiffness and stress

$$E_{ijkl} = (1-D)E^{0}_{ijkl}; \quad \dot{E}_{ijkl} = -\dot{D}E^{0}_{ijkl} \tag{108a, b}$$

$$\dot{\lambda} = \dot{D}; \quad \bar{M}_{ijkl} = -E^{0}_{ijkl}; \quad \bar{m}_{ij} = -E^{0}_{ijlk}\varepsilon_{kl} \tag{108c, d, e}$$

that are needed to obtain the tangential stiffness. Additionally, one can obtain the various terms of the formulation considered as a damage formulation (in the terminology of this paper, Section 6.2) with the damage variable $\bar{\mathscr{D}}_* = \text{scalar} = D$, as $\partial E_{ijkl}/\partial\bar{\mathscr{D}}_* = -E^{0}_{ijlk}$ and $\mathscr{M}_* = \text{scalar} = 1$. Note that from this expression and using (66b), one can obtain $\bar{\mathbf{M}}$ in (108d).

Now, the following loading function is chosen:

$$F = f[\bar{w}^{0}, D] - k[D] \quad \text{where} \quad \bar{w}^{0} = \tfrac{1}{2}\varepsilon_{ij}E^{0}_{ijkl}\varepsilon_{kl}. \tag{109a, b}$$

Note that $\bar{w}^{0}$ is the strain-based undamaged free energy. Although in (109a) $\bar{w}^{0}$ has been used for convenience, the standard free energy may be written as $\bar{w} = (1-D)\bar{w}^{0}$ and the undamaged stress-based free energy as $w^{0} = \sigma_{ij}C^{0}_{ijkl}\sigma_{kl}/2 = (1-D)^{2}\bar{w}^{0}$, and therefore the cases in which F is a function of any of them are also included in (109a). Note also that the loading function (109a) guarantees full associativity since it is expressed in terms of the mechanical free energy.

From F, the gradients in stiffness and strain space can be developed

$$\bar{N}_{ijkl} = \frac{\partial F}{\partial(-\frac{1}{2}\varepsilon_{ij}\varepsilon_{kl})} = -\frac{\partial f}{\partial \bar{w}^0} E^0_{ijkl}; \quad \bar{n}_{ij} = -\bar{N}_{ijkl}\varepsilon_{kl} = \frac{\partial f}{\partial \bar{w}^0} E^0_{ijkl}\varepsilon_{kl} \tag{110a, b}$$

where the partial derivative is for constant D. Again, these expressions suffice in this type of formulation to obtain the tangential stiffness; however, one can additionally obtain the gradient of F in the damage space $\mathcal{N}_* = \partial F/\partial(-\mathcal{Y}_*)$ (where in this case $-\mathcal{Y}_* = \bar{w}^0$), as $\bar{\mathcal{N}}$ = scalar $= \partial f/\partial \bar{w}^0$. Because of its scalar character, $\bar{\mathcal{N}}$ is always proportional to $\bar{\mathcal{M}}$ which implies full associativity. Note also that, by introducing this expression of $\bar{\mathcal{N}}$ into (69b), one obtains the same expression (110a) for $\bar{\mathbf{N}}$.

From F one can also obtain the hardening function $\bar{H} = -\partial F/\partial\lambda$ (partial derivative for constant strain), where according to (108c) λ can be replaced by D, yielding

$$\bar{H} = -\frac{\partial f[\bar{w}^0, D]}{\partial D} + \frac{\partial k(D)}{\partial D}. \tag{111}$$

Since $\bar{w}^0 = \varepsilon_{ij}E^0_{ijkl}\varepsilon_{kl}/2$ and E^0_{ijkl} = constant, the partial derivatives for constant strain in this expression can be equivalently taken as for constant $\bar{w}^0$.

By introducing $\bar{\mathbf{n}}$, $\bar{\mathbf{M}}$ and $\bar{H}$ into (49a), and after some manipulation [e.g. replacing $\mathbf{E}^0 = \mathbf{E}/(1-D)$, etc.], one obtains directly the tangent operator in eqn (107) with

$$\bar{h} = \frac{(1-D)^2}{\partial f/\partial \bar{w}^0}\left(\frac{\partial k}{\partial D} - \frac{\partial f}{\partial D}\right). \tag{112}$$

It is easy to verify that all the asociated models in the previous section are particular cases of this general formulation. For the model by Simo–Ju strain-based (Section 7.2), one has

$$f[\bar{w}^0, D] = \sqrt{2\bar{w}^0}; \quad \frac{\partial f}{\partial D} = 0; \quad \frac{\partial f}{\partial \bar{w}^0} = \frac{1}{\sqrt{2\bar{w}^0}} \tag{113a, b, c}$$

$$\text{and} \quad \bar{h} = (1-D)^2\sqrt{2\bar{w}^0}\,\frac{\partial k}{\partial D}. \tag{114}$$

By replacing this expression of $\bar{h}$ into (107), one obtains the same as (82) with the substitutions $\sigma^0_{ij} = \sigma_{ij}/(1-D)$, $\bar{\tau} = \sqrt{2\bar{w}^0}$ and $\bar{r} = k$.

For the model by Simo–Ju stress-based (Section 7.2), one has

$$f[\bar{w}^0, D] = (1-D)\sqrt{2\bar{w}^0}; \quad \frac{\partial f}{\partial D} = -\sqrt{2\bar{w}^0}; \quad \frac{\partial f}{\partial \bar{w}^0} = \frac{1-D}{\sqrt{2\bar{w}^0}} \tag{115a, b, c}$$

$$\text{and} \quad \bar{h} = (1-D)\sqrt{2\bar{w}^0}\left(\frac{\partial k}{\partial D} + \sqrt{2\bar{w}^0}\right). \tag{116}$$

By replacing this expression of $\bar{h}$ into (107), one obtains the same as (90) with the substitutions $\tau = (1-D)\sqrt{2\bar{w}^0}$ and $r = k$, and the change of derivative $\partial/\partial\bar{D} = (1-D)^2\,\partial/\partial D$.

For the models by Ju (Section 7.3) and Benallal *et al.* [which is equivalent (Section 7.4)] one has

$$f[\bar{w}^0, D] = \bar{w}^0; \quad \frac{\partial f}{\partial D} = 0; \quad \frac{\partial f}{\partial \bar{w}^0} = 1 \tag{117a, b, c}$$

$$\text{and} \quad \bar{h} = (1-D)^2\frac{\partial k}{\partial D}. \tag{118}$$

By replacing this expression of $\bar{h}$ into (107), one obtains the same as (94) with $\sigma^0_{ij} = \sigma_{ij}/(1-D)$ and $\bar{r}'' = k$.

Finally, for the model by Neilsen and Schreyer (Section 7.5), one has

$$f[\bar{w}^0, D] = (1-D)^2\bar{w}^0\,; \quad \frac{\partial f}{\partial D} = -2(1-D)\bar{w}^0\,; \quad \frac{\partial f}{\partial \bar{w}^0} = (1-D)^2 \qquad (119\text{a, b, c})$$

$$\text{and} \quad \bar{h} = \frac{\partial k}{\partial D} - \varepsilon_{pq}\sigma_{pq}. \qquad (120)$$

By replacing this expression of $\bar{h}$ into (107), one obtains the same tangent operator as (106) with $g = k$.

9. SUMMARY AND CONCLUSIONS

(1) The existing literature on stiffness degradation does not provide a general constitutive framework for this type of model. Although some attempts have been made in this sense, most of the proposals published recently represent particular formulations, each using its different terminology and notation, and most of them combining stiffness degradation with other effects, which makes it difficult to isolate, analyse and understand the properties of this type of model.

(2) The elastic-degrading material (material with degradation of elastic stiffness) can be formulated using the concept of loading surface, with a set of descriptors which follow those of classical elastoplasticity, in which the plastic strains are replaced by degrading strains, and the initial elastic stiffness (or compliance) is replaced by the current (secant) one. The plasticity-like formulation has to be supplemented with equations for the evolution of the compliance (or stiffness) itself. The full set of new concepts can be defined in the compliance (or stiffness) space, in a similar way as those normally considered in the strain space: degradation rule, degrading dissipation rate, conjugate force, normal to the loading surface and associativity. Simple relations emerge between the new concepts and their strain space counterparts. Similarly to plasticity, the whole formulation can be developed one-to-one either in the strain or stress space and the dual terms are directly related to each other.

(3) Rather than directly, the evolution of compliance (or stiffness) can be more efficiently formulated through a reduced set of damage variables which completely define the current state of degradation in the material. Elastic-damage formulations can be developed similarly as before by considering a damage space (similar to the compliance or the strain spaces), and an analogous set of concepts in that space. Again, the formulation can be developed either based on stress or strains, and relations between dual terms as well as with their compliance space or strain space-based counterparts can be readily obtained.

(4) The whole set of relations presented throughout the paper and summarized in the tables in the Appendix, constitutes a general unified framework for the description of materials with degradation of elastic properties. This framework incorporates most of the preceding theories and links them together with a common plasticity-like terminology and relatively little recourse to abstract thermodynamic concepts.

(5) Some of the most relevant formulations found in the literature for scalar damage of the traditional $(1-D)$ type have been considered. It has been shown that, restricting attention to the behavior of these models with regard to elastic degradation (i.e. disregarding aspects such as plasticity, positive–negative projections, etc.) all these models can be described easily within the framework provided by the unified theory; their properties can be analysed, and the similarities and differences between them can be determined and understood.

(6) Furthermore, the unified theory offers the possibility of a general formulation for associated $(1-D)$ scalar damage formulations of both stress- and strain-based type, with a single general derivation and a unique expression for the corresponding tangential operator. This formulation has been developed and presented, and it has been shown to encompass

all the models of this type extracted from current literature. A systematic review of other types of damage models such as scalar non-$(1-D)$, vectorial and tensorial, developed also within the framework of the same unified theory will be presented in a sequel paper.

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APPENDIX: SUMMARY OF EQUATIONS

Table 1. Elastic-degrading materials, stress- and strain-based formulations

Secant relation Formulation	$\varepsilon = \mathbf{C}:\sigma$ Stress-based $F(\sigma, \mathbf{p})$	$\sigma = \mathbf{E}:\varepsilon$ Strain-based $F(\varepsilon, \bar{\mathbf{p}})$
	Strain space	Stress space
Flow rule	$\dot{\varepsilon}_d = \dot{\lambda}\mathbf{m}\left(\text{optional } \mathbf{m} = \frac{\partial Q}{\partial \sigma}\right)$	$\dot{\sigma}_d = \dot{\lambda}\bar{\mathbf{m}}\left(\text{optional } \bar{\mathbf{m}} = \frac{\partial \bar{Q}}{\partial \varepsilon}\right)$
Gradient/hardening	$\mathbf{n} = \left.\frac{\partial F}{\partial \sigma}\right\|_{\lambda} \quad H = -\left.\frac{\partial F}{\partial \lambda}\right\|_{\sigma}$	$\bar{\mathbf{n}} = \left.\frac{\partial F}{\partial \varepsilon}\right\|_{\lambda} \quad \bar{H} = -\left.\frac{\partial F}{\partial \lambda}\right\|_{\varepsilon}$
Tangent stiffness	$\mathbf{E}_t = \mathbf{E} - \frac{\mathbf{E}:\mathbf{m}\otimes\mathbf{n}:\mathbf{E}}{H+\mathbf{n}:\mathbf{E}:\mathbf{m}}$	$\mathbf{E}_t = \mathbf{E} + \frac{\bar{\mathbf{m}}\otimes\bar{\mathbf{n}}}{\bar{H}}$
Tangent compliance	$\mathbf{C}_t = \mathbf{C} + \frac{\mathbf{m}\otimes\mathbf{n}}{H}$	$\mathbf{C}_t = \mathbf{C} - \frac{\mathbf{C}:\bar{\mathbf{m}}\otimes\bar{\mathbf{n}}:\mathbf{C}}{\bar{H}+\bar{\mathbf{n}}:\mathbf{C}:\bar{\mathbf{m}}}$
Relations	$\mathbf{m} = -\mathbf{C}:\bar{\mathbf{m}}$ $\mathbf{n} = \mathbf{C}:\bar{\mathbf{n}}$ $H = \bar{H}+\bar{\mathbf{n}}:\mathbf{C}:\bar{\mathbf{m}}$	$\bar{\mathbf{m}} = -\mathbf{E}:\mathbf{m}$ $\bar{\mathbf{n}} = \mathbf{E}:\mathbf{n}$ $\bar{H} = H+\mathbf{n}:\mathbf{E}:\mathbf{m}$
Associativity	$\mathbf{n}\parallel\mathbf{m} \quad (\text{or } Q = F)$	$\bar{\mathbf{n}}\parallel\bar{\mathbf{m}} \quad (\text{or } \bar{Q} = F)$
	Compliance space	Stiffness space
Degradation rule	$\left[\dot{\mathbf{C}} = \dot{\lambda}\mathbf{M}\left(\text{optional } \mathbf{M} = \frac{\partial Q'}{\partial(-\mathbf{Y})}\right)\right]$	$\left[\dot{\mathbf{E}} = \dot{\lambda}\bar{\mathbf{M}}\left(\text{optional } \bar{\mathbf{M}} = \frac{\partial \bar{Q}'}{\partial(-\bar{\mathbf{Y}})}\right)\right]$
Thermodynamic force	$\left[(-\mathbf{Y}) = \left.\frac{\partial w}{\partial \mathbf{C}}\right\|_{\sigma} = \frac{1}{2}\sigma\otimes\sigma\right]$	$\left[(-\bar{\mathbf{Y}}) = -\left.\frac{\partial w}{\partial \mathbf{E}}\right\|_{\varepsilon} = -\frac{1}{2}\varepsilon\otimes\varepsilon\right]$
Generalized gradient	$\left[\mathbf{N} = \left.\frac{\partial F}{\partial(-\mathbf{Y})}\right\|_{\lambda}\right]$	$\bar{\mathbf{N}} = \left.\frac{\partial F}{\partial(-\bar{\mathbf{Y}})}\right\|_{\lambda}$
Flow/degradation rules	$\mathbf{m} = \mathbf{M}:\sigma$	$\bar{\mathbf{m}} = \bar{\mathbf{M}}:\varepsilon$
Gradient/gen. gradient	$\mathbf{n} = \mathbf{N}:\sigma$	$\bar{\mathbf{n}} = -\bar{\mathbf{N}}:\varepsilon$
Relations	$\mathbf{M} = -\mathbf{C}:\bar{\mathbf{M}}:\mathbf{C}$ $\mathbf{N} = -\mathbf{C}:\bar{\mathbf{N}}:\mathbf{C}$	$\bar{\mathbf{M}} = -\mathbf{E}:\mathbf{M}:\mathbf{E}$ $\bar{\mathbf{N}} = -\mathbf{E}:\mathbf{N}:\mathbf{E}$
Associativity	$\mathbf{N}\parallel\mathbf{M} \quad (\text{or } Q' = Q = F)$	$\bar{\mathbf{N}}\parallel\bar{\mathbf{M}} \quad (\text{or } \bar{Q}' = \bar{Q} = F)$

Table 2. Damage-elastic-degrading models, compliance formulation

	Compliance damage space
Formulation	Stress-based $F(\boldsymbol{\sigma}, \mathbf{p})$
Secant relation	$\boldsymbol{\varepsilon} = \mathbf{C}(\mathbf{C}_0, \mathscr{D}) : \boldsymbol{\sigma}$
Damage rule	$\dot{\mathscr{D}} = \dot{\lambda}\mathscr{M} \quad \left(\text{optional } \mathscr{M} = \dfrac{\partial Q''}{\partial(-\mathscr{Y})}\right)$
Thermodynamic force	$(-\mathscr{Y}) = \left.\dfrac{\partial w}{\partial \mathscr{D}}\right\|_{\sigma} = (-\mathbf{Y}) :: \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}}$
Generalized gradient	$\mathscr{N} = \left.\dfrac{\partial F}{\partial(-\mathscr{Y})}\right\|_{\lambda}$
Degradation/damage rules	$\mathbf{M} = \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} * \mathscr{M}$
Gen. gradient/gen. gradient	$\mathbf{N} = \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} * \mathscr{N}$
Relations	$\mathscr{M} = -\left(\dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} :: \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}}\right)^{-1} * \left(\dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} :: \left(\mathbf{C} : \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} : \mathbf{C}\right)\right) * \bar{\mathscr{M}}$ $\mathscr{N} = -\left(\dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} :: \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}}\right)^{-1} * \left(\dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} :: \left(\mathbf{C} : \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} : \mathbf{C}\right)\right) * \bar{\mathscr{N}}$
Associativity	$\mathscr{N} \parallel \mathscr{M} \quad (\text{or } Q'' = Q' = Q = F)$

Table 3. Damage-elastic-degrading models, stiffness formulation

	Stiffness damage space
Formulation	Stress-based $F(\boldsymbol{\varepsilon}, \bar{\mathbf{p}})$
Secant relation	$\boldsymbol{\sigma} = \mathbf{E}(\mathbf{E}_0, \bar{\mathscr{D}}) : \boldsymbol{\varepsilon}$
Damage rule	$\dot{\bar{\mathscr{D}}} = \dot{\lambda}\bar{\mathscr{M}} \quad \left(\text{optional } \bar{\mathscr{M}} = \dfrac{\partial \bar{Q}''}{\partial(-\bar{\mathscr{Y}})}\right)$
Thermodynamic force	$(-\bar{\mathscr{Y}}) = -\left.\dfrac{\partial w}{\partial \bar{\mathscr{D}}}\right\|_{\varepsilon} = (-\bar{\mathbf{Y}}) :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}}$
Generalized gradient	$\bar{\mathscr{N}} = \left.\dfrac{\partial F}{\partial(-\bar{\mathscr{Y}})}\right\|_{\lambda}$
Degradation/damage rules	$\bar{\mathbf{M}} = \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} * \bar{\mathscr{M}}$
Gen. gradient/gen. gradient	$\bar{\mathbf{N}} = \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} * \bar{\mathscr{N}}$
Relations	$\bar{\mathscr{M}} = -\left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}}\right)^{-1} * \left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} :: \left(\mathbf{E} : \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} : \mathbf{E}\right)\right) * \mathscr{M}$ $\bar{\mathscr{N}} = -\left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}}\right)^{-1} * \left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathscr{D}}} :: \left(\mathbf{E} : \dfrac{\partial \mathbf{C}}{\partial \mathscr{D}} : \mathbf{E}\right)\right) * \mathscr{N}$
Associativity	$\bar{\mathscr{N}} \parallel \bar{\mathscr{M}} \quad (\text{or } \bar{Q}'' = \bar{Q}' = \bar{Q} = F)$